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Preface 2012

ISAMA 2012 takes place in Chicago, Illinois and is co-hosted by Steve Luecking of De Paul University and Robert Krawczyk of the Illinois Institute of Technology. The invited speakers are Robert Bosch, Oberlin College; Gabrielle Meyer, University of Wisconsin; and Jeanne Gang and Mark Schendel, Studio Gang Architects, Chicago.

This conference is the twentieth year anniversary of the first Art and Mathematics Conference (AM 92) organized by Nat Friedman at SUNY-Albany in June, 1992. This conference was followed by annual conferences AM93-AM97 at Albany and AM 98 at the University of California, Berkeley, co-organized with Carlo Sequin. ISAMA was founded by Nat Friedman in 1998 along with the ISAMA publication Hyperseeing co-founded with Ergun Akleman, Texas A&M, in 2006. In addition, the Art/Math movement has taken off with the formation of many additional conferences and organizations. In particular, we mention the very successful conference Bridges organized by Reza Sarhangi in 1998 and the excellent Bridges Proceedings. The significance of the art/math movement is now recognized internationally and in particular by the extensive art/math exhibit at the annual Joint Mathematics Meeting of the American Mathematical Society and the Mathematical Association of America organized by Robert Fathauer.

In this volume there are a range of papers relating the arts, mathematics, and architecture. We wish to thank the authors for their participation in ISAMA 2012-you are the conference. Our purpose is to come together to share information and discuss common interests. Hopefully new ideas and partnerships will emerge which can enrich interdisciplinary education.

ISAMA 2012 Organizing Committee

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Bourgoin's Twelve-Pointed Star Polygon Designs in Cairo

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Abstract

This paper investigates the twelve-pointed star polygon designs illustrated in Bourgoin's *Arabic Geometrical Pattern and Design* and found as ornament on extant Islamic monuments in Cairo. Fifteen highly symmetric patterns from the Mamluk period, exhibiting either four-fold or six-fold rotational symmetry with multiple mirror reflections, were identified. Seven of the designs may be classified as *p*4*m* patterns while the remaining eight may be classified as *p*6*m*. Seven of the patterns contain only large twelve-pointed stars, while the other eight had twelve-stars in combination with seven-, eight-, nine- or sixteen-pointed stars.

Introduction

Jules Bourgoin was a 19th century French architect, who, after travelling and spending much time in Egypt, became fascinated with the Islamic art there. In 1879, he published *Arabic Geometrical Pattern and Design* [1], a rich source of over 200 Islamic patterns that were based upon his sketches of the ornament found on historic monuments in Cairo and Damascus. Seventy (or more than a third) of Bourgoin's drawings in [1] contain large twelve-pointed star polygons, either alone or in combinations with other *n*-star polygons, where n = 6, 7, 8, 9, 10, 15, 18 and 20. Could some of these twelve-pointed star designs still be found on existing Islamic monuments in Cairo? By searching photo archives, including *Pattern in Islamic Art: The Wade Photo Archive* [2], the author set out to answer this question. In addition, we also investigated whether the designs were documented in the *Topkapı* or *Tashkent Scrolls,* two sources of Islamic architectural and ornamental design sketches dating from the 15th or 16th century.

The Twelve-Star Designs

Fifteen of Bourgoin's twelve-star designs have been identified on historic monuments dating from the Mamluk period (or between the 13th and 16th centuries). All are highly symmetric, exhibiting either four-fold or six-fold rotational symmetry with multiple mirror reflections. Seven of the designs may be classified as belonging to the *p*4*m* crystallographic symmetry group, while the remaining eight may be classified as *p*6*m* patterns. Seven of the designs are comprised of only large twelve-pointed stars, while the other eight are twelve-stars in combination with *regular* (or *nearly-regular*) seven-, eight-, nine- or sixteen-pointed stars. None of Bourgoin's 12-star patterns are also documented in the *Topkapı Scroll* which has only six repeat units that can generate planar 12-star designs (thus excluding the sketches for muqarnas), although three repeat units do generate patterns that closely match the designs in Bourgoin's Plates 78, 118 and 126. Likewise, the *Tashkent Scrolls* do not have any repeat units that produce any of Bourgoin's 12-star designs. However, there is one 9- and 12-pointed star Tashkent repeat unit that does produce a design very similar to Bourgoin's Plate 120. These four patterns are addressed in more detail in the Discussion section at the end of this paper).

In the next section, a brief description and photograph of each of the patterns is provided, along with one possible repeat unit superimposed on a portion of Bourgoin's plates, starting first with the *p*6*m* patterns.

The *p6m* Patterns

All of the examples of Bourgoin's designs in this section on *p*6*m* patterns have *regular* 12-stars that may be categorized as {12/5} star polygons. That is, the shape of the stars may be reproduced by connecting with line segments every fifth of twelve equally-spaced points on a circle (**Figure 0a)**. By erasing some segments, we may form a more decorative star polygon (**Figure 0b)**, which is identical to the skeletal version (that is, that do not consider interlacing effects and so on) of the 12-star designs exemplified in **Figures 1a – 8a**. A possible repeat unit for all of these *p*6*m* patterns is a *regular* hexagon. That is, the entire pattern for each example may be reproduced by replicating its repeat unit through the use of symmetry operations, such as reflection across an edge of the repeat unit or by rotation about a vertex. Each repeat unit has been selected so the largest of the star polygons in each design is at the center of the repeat unit. Thus, the center of the hexagon (and also the center of each design's largest star) serves as a 6-fold rotocenter with multiple mirror axes possible through this point for all the *p*6*m* designs. The reflection mirrors may be imagined to lie on line segments formed by connecting opposite vertices (shown with dashed line segments) or midpoints of sides (shown with dotted line segments) in **Figure 0c**.

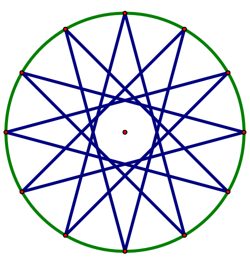


Figure 0a. *A {12/5} star polygon*

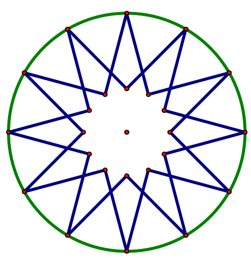


Figure 0b. *A more decorative {12/5} star*

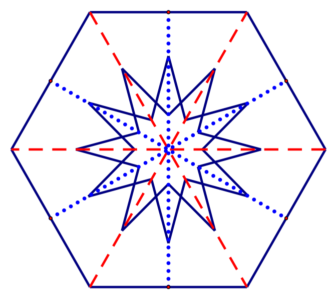


Figure 0c. *A hexagonal repeat unit showing possible mirror axes of a p6m pattern*

Bourgoin's Plates 76 and 82 (only 12-stars)

The first two *p*6*m* designs may be found on stone inlay mosaics dating to1347 and found in the Masjid al-Aqsunqur, also known as the Blue Mosque due to the color of tiles installed in the 17th century. It is considered to be an exceptional example of early Mamluk religious architecture [3]. Cropped versions of Wade's catalog numbers *EGY 1619* and *EGY 1621* are shown in **Figures 1a** and **2a** (on the following page). Copies of a small portion of Bourgoin's Plates 76 and 82 (**Figures 1b** and **2b**, respectively) show the hexagonal repeat units each with a large 12-star at the center.

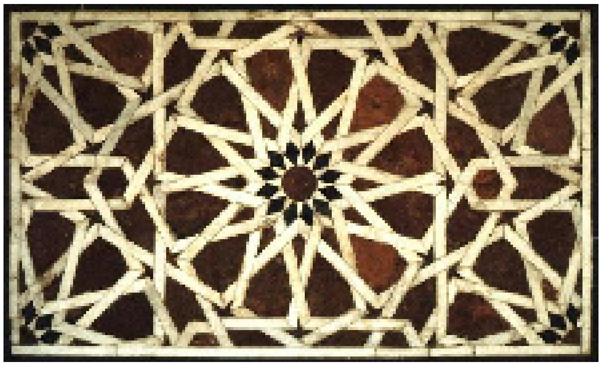


Figure 1a. *A portion of a stone inlay mosaic in the Masjid al-Aqsunqur, EGY 1619, dated 1347*

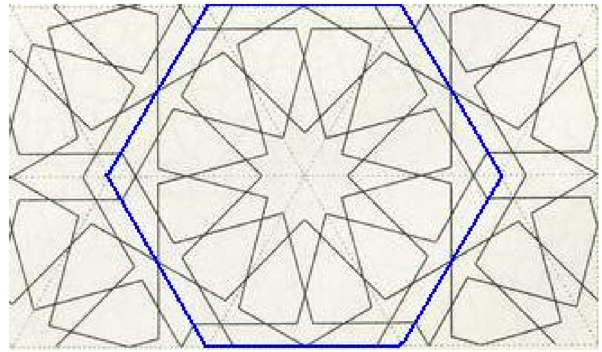


Figure 1b. *A portion of Bourgoin's Plate 76, cropped by author, with a repeat unit outlined*

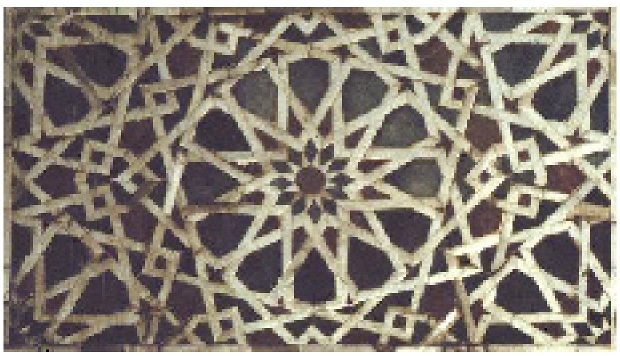


Figure 2a. *A portion of a stone inlay mosaic in the Masjid al-Aqsunqur, EGY 1621, dated 1347*

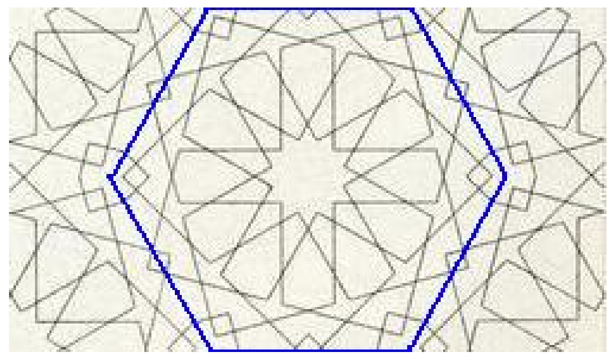


Figure 2b. *A portion of Bourgoin's Plate 82, cropped by author, with a repeat unit outlined*

Bourgoin's Plate 99

The third *p*6*m* design may be found on a carved masonry stone relief in the Amir Taz Palace, one of the most well-known Mamluk Palaces remaining in historic Cairo. It was built in 1352 by Amir Taz to celebrate his marriage to Khwand Zahra, the daughter of Sultan Al-Nasir Mohammad [4]. A cropped version of Wade's catalog number *EGY 0526* is provided in **Figure 3a**, along with a possible repeat unit showing a 12-star at its center in **Figure 3b**. Note that two small line segments forming an "X" (just to the right of the hexagon and equally spaced between the two 12-stars on the upper and lower edges of **Figure 3b)** do not exist as part of the actual stone relief.

Bourgoin's Plate 68 (only 12-stars)

A fourth *p*6*m* design may be found as a pierced stone latticework screen, also at the Amir Taz Palace, dating to1352. A cropped version of Wade's catalog number *EGY 0517* is provided in **Figure 4a** on the following page, along with a possible hexagonal repeat unit shown in **Figure 4b**.

Figure 3a. *A portion of carved masonry/stone relief in the Amir Taz Palace, EGY 0526, dated 1352*

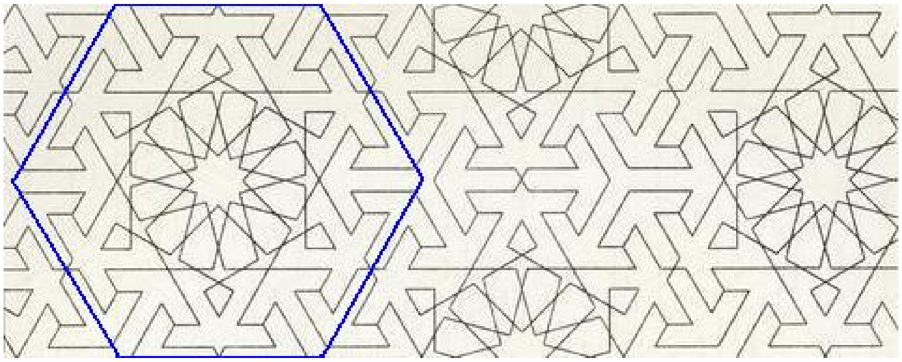


Figure 3b. *A portion of Bourgoin's Plate 99, cropped by author, with a repeat unit outlined*

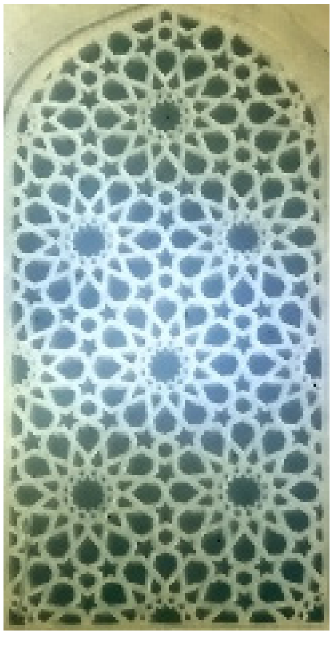


Figure 4a. *A pierced stone lattice work in the Amir Taz Palace, EGY0517, dated 1352*

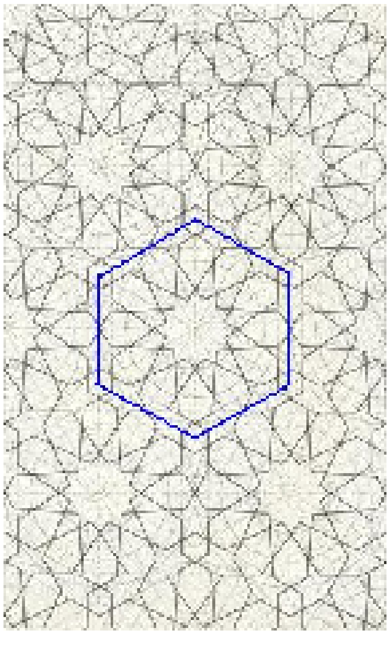


Figure 4b. *A portion of Bourgoin's Plate 68, cropped by the author, with a repeat unit outlined*

Bourgoin's Plate 72 (only 12-stars)

The fifth *p*6*m* design may be found as a wood panel inlaid with mother of pearl at the Masjid al-Mu'ayyad, built between 1415 and 1421 by the Mamluk sultan, al-Mu'ayyad Sayf ad-Din Shaykh, one of the great patrons of architecture in Cairo. The madrasa within the mosque became one of the most prominent academic institutions in 15th century Cairo, while the sanctuary was one of the most richly decorated of its time [5]. A cropped version of Wade's catalog number *EGY 1208* is provided in **Figure 5a**, along with a possible hexagonal repeat unit, shown in **Figure 5b**.

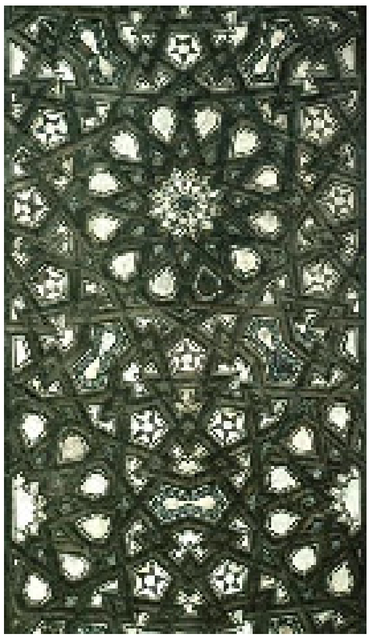


Figure 5a. *A wood panel with inlay in the Masjid al-Mu'ayyad, dated 1420*

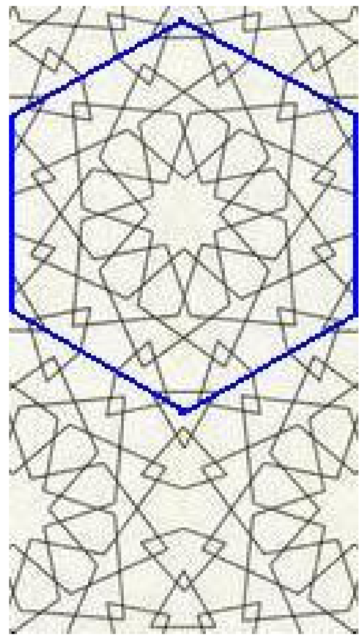
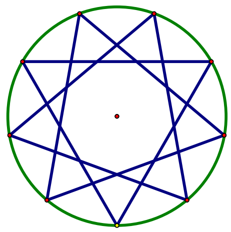


Figure 5b. *A portion of Bourgoin's Plate 72, cropped by author, with a repeat unit outlined*

Bourgoin's Plate 124 (12- and 9-stars)

The first five *p*6*m* designs discussed so far consist solely of *regular* twelve-pointed star polygons. The next three designs have combinations of regular 12-stars and "nearly regular" nine-stars. This sixth pattern, consisting of both {12/5} star polygons and {9/3} star figures, may be found as a carved wood panel on a minbar in the al-Ghuriyya Complex, built by Sultan al Ghuri between 1503 and 1505. The complex includes a mosque, madrasa, *khanqah* (monastery), mausoleum and *sabil-kuttab* (a Qur'anic school for boys on the top floor with a drinking fountain on the ground floor). Ironically, Al Ghuri died in a military campaign against the Ottomans outside Aleppo. And since his body was never found, he wasn't buried in his mausoleum on which he had spent a fortune [6].

To create the {9/3} star figure that appears in this design, connect with line segments every third of nine equally-spaced points on a circle (**Figure 6a** on the following page). By erasing some segments, we may form a more decorative star polygon (**Figure 6b**).



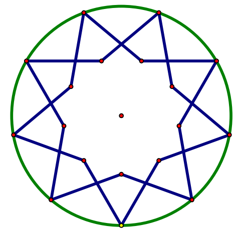


Figure 6a. *A {9/3} star figure* **Figure 6b.** *A more decorative {9/3} star*

A cropped version of Wade's catalog number *EGY 1722* is provided in **Figure 6c**, along with a possible hexagonal repeat unit, shown in **Figure 6d**. Note that a third of the 9-star motif, which is centered on each vertex of the repeat unit, appears within the hexagon.

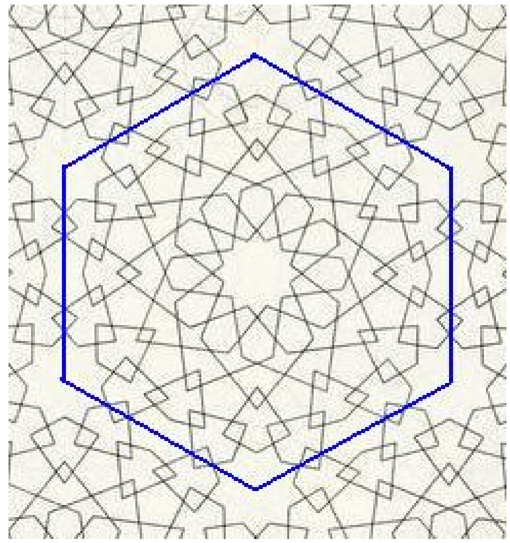
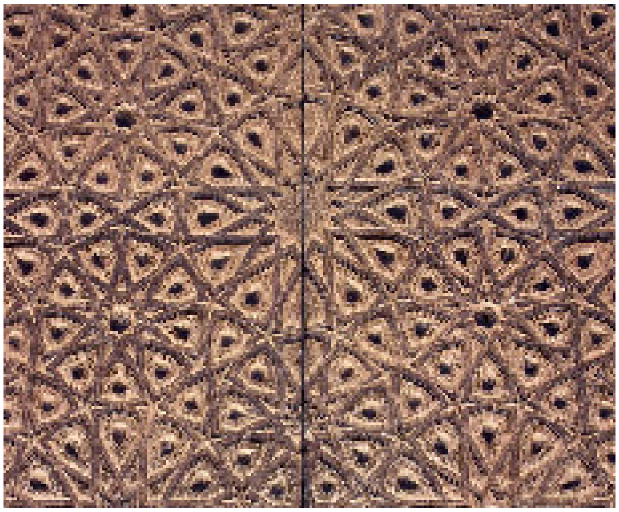


Figure 6c. *A wood panel on the minbar of the al-Ghuriyya Complex, EGY1722, dated 1504*

Figure 6d. *A portion of Bourgoin's Plate 124, cropped by author, with a repeat unit outlined*

Bourgoin's Plates 120 and 121 (12- and 9-stars)

The last two *p*6*m* designs also consist of regular twelve-pointed star polygons and "nearly regular" nine-stars. Interestingly, they share the exact same structure; the designs of Plate 120 are made from straight line segments, thus forming both {12/5} star polygons and {9/3} star figures, while for Plate 121, arcs are used to create the "rounded" version of Plate 120. Both examples are found on copper doors, the first at the *zawiya* (shrine) of al-Sultan Baybars II dating to 1310 (**Figure 7a** on the following page), and the second at the Masjid al-Sultan al-Nasir Hasan, dating to 1363 (**Figure 8a**). Possible hexagonal repeat units, with 12-stars at each center, are shown in **Figures 7b** and **8b**. Note, again, that a third of the 9-star motif, which is centered on each vertex of the hexagon, appears within the repeat unit. A Euclidean (compass and straightedge) reconstruction of the Plate 120 pattern may be found in [7].



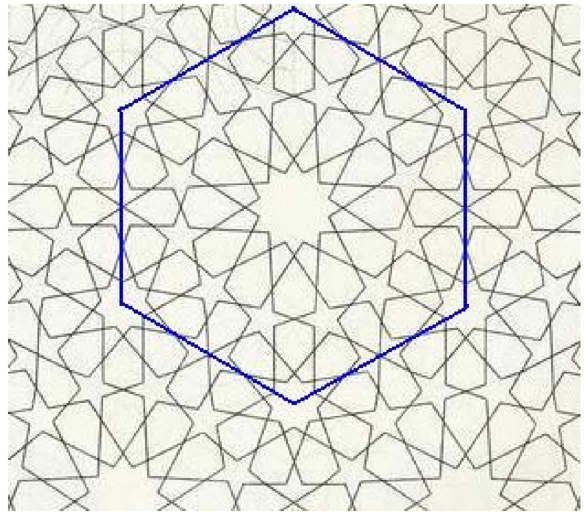
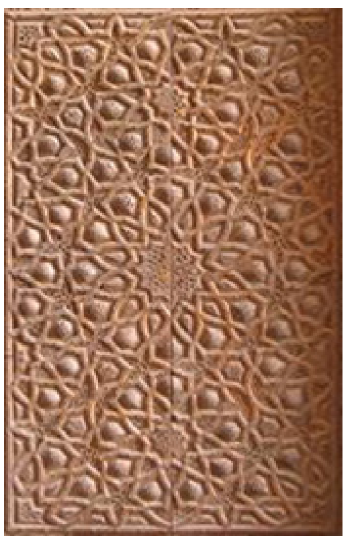


Figure 7a. *A bronze door of the zawiya of al-Sultan Baybars II, dated 1310*

Figure 7b. *A portion of Bourgoin's Plate 120, cropped by author, with the repeat unit outlined*



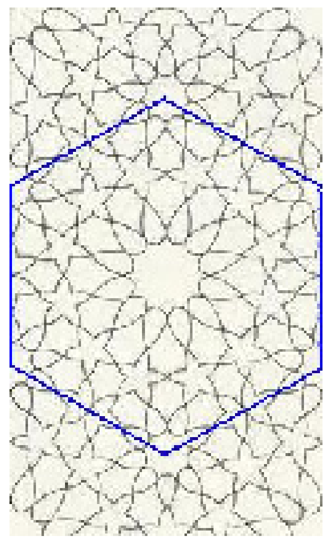


Figure 8a. *A bronze door of the Masjid al-Sultan al-Nasir Hasan, dated 1363*

Figure 8b. *A portion of Bourgoin's Plate 121, cropped by author, with the repeat unit outlined*

The *p*4*m* Patterns

In this next section, the examples of Bourgoin's designs show four-fold symmetry and thus may be classified as *p*4*m* patterns. They have *regular* 12-stars that are {12/5} star polygons, or {12/4} or {24/9} star figures. That is, the shape of the stars may be reproduced by connecting with line segments every fourth of twelve equally-spaced points on a circle, or every ninth of twenty-four equally-spaced points on a circle, respectively. A {12/4} star figure is shown in **Figures 9a** and **9c**. After erasing some line segments, more decorative stars are formed as shown in **Figures 9b** and **9d**. But then to get the desired 12-star erase half of the points as shown in **Figure 9e**.

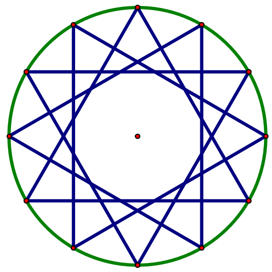


Figure 9a. *A {12/4} star figure*

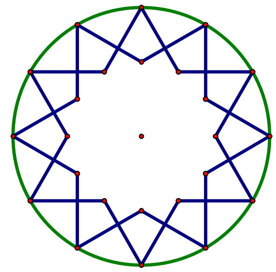


Figure 9b. *A more decorative {12/4} star*

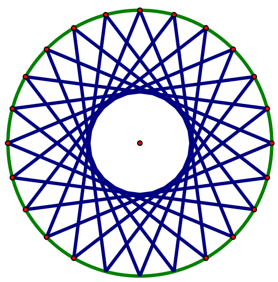


Figure 9c. *A {24/9} star figure*

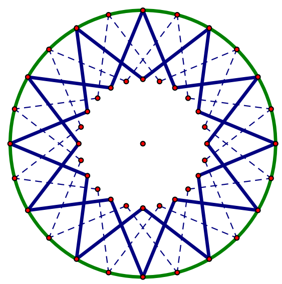


Figure 9d. *A more decorative {24/9} star*

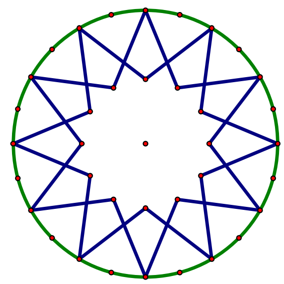


Figure 9e. *The {24/9} star with half of the points erased*

A possible repeat unit for all of the *p*4*m* patterns is a square. Just as with the hexagonal repeat unit for the *p*6*m* designs, the entire pattern for each *p*4*m* example may be reproduced by replicating the repeat unit for each by reflecting across an edge of the repeat unit or by rotating about a vertex. Each repeat unit has been selected so the largest of the star polygons in each design is at the center of the repeat unit.
Thus, for all the *p*4*m* designs, the center of the square (or the center of the design's largest star) serves as a 4-fold rotocenter with multiple mirror axes possible through this point. The reflection mirrors may be imagined to lie on line segments formed by connecting opposite vertices shown with dashed line segments or midpoints of sides shown with dotted line segments in **Figures 9f** for a {12/5} star, **9g** for a {12/4} star and **9h** for a {24/9} star (with half of its points erased).

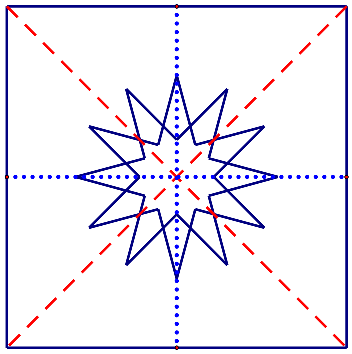


Figure 9f. *A square repeat unit showing possible mirror axes on a {12/5} star*

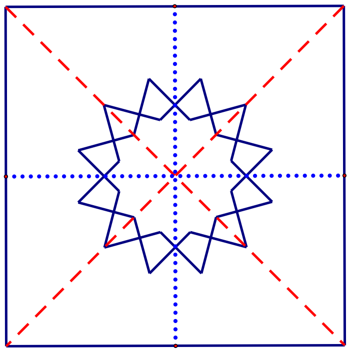


Figure 9g. *A square repeat unit showing possible mirror axes on a {12/4} star*

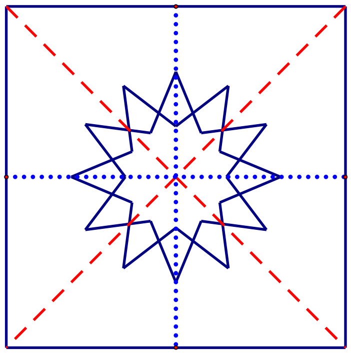


Figure 9h. *A square repeat unit showing possible mirror axes on a {24/9} star (with half of the points erased)*

Bourgoin’s Plate 94 (only 12-stars)

The first of the *p*4*m* patterns may be found as a painted wood panel on a minbar in the Masjid al-Salih Tala’i’. Although this masjid – the last of the Fatimid buildings – was originally built by Vizier *Salih Tala’i’* in 1160, its minbar is of Mamluk origin, having been added to the mosque in 1299. It is also the 2nd oldest Mamluk minbar in Cairo, with the oldest one being in the Masjid Ibn Tulun. Behind the minbar in the *qibla* wall is the earliest extant example in Cairo of a “wind catcher” (*malqaf*), which is a framed rectangular opening connected to a shaft going to the roof where a sloping lid opens to the north to catch the breezes [8]. The 12-star in this design is a {12/4} star figure (**Figure 9b**). A cropped version of Wade’s catalog number *EGY 1527* is provided in **Figure 9i**, along with a possible square repeat unit, shown in **Figure 9j**.

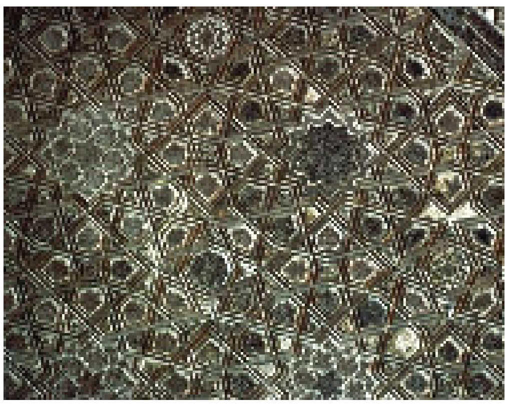


Figure 9i. *A wood panel on a minbar of the Masjid al-Salih Tala’i’, dated 1299*

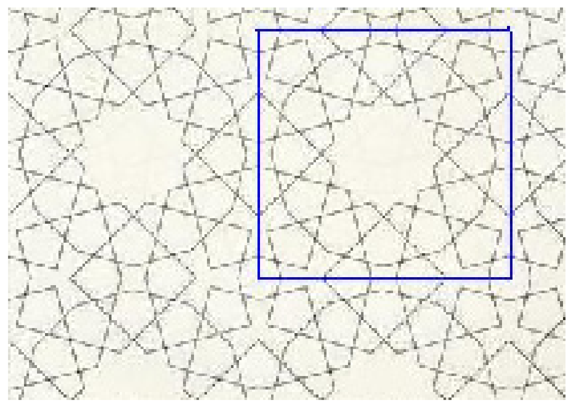


Figure 9j. *A portion of Bourgoin’s Plate 94, cropped by author, with the repeat unit outlined*

Bourgoin’s Plate 77 (only 12-stars)

The second of the *p*4*m* patterns, consisting solely of {12/5} star polygons, may also be found as a carved and painted wood panel on a minbar of the Masjid al-Nasir, dating to 1304. A cropped version of Wade’s catalog number *EGY 0731* is shown in **Figure 10a**. A possible square repeat unit is shown highlighted on a portion of Bourgoin’s Plate 77 with the 12-star at the center (**Figure 10b**).

Figure 10a. *A wood panel on the minbar of the Masjid al-Nasir, EGY 0731, dated 1304*

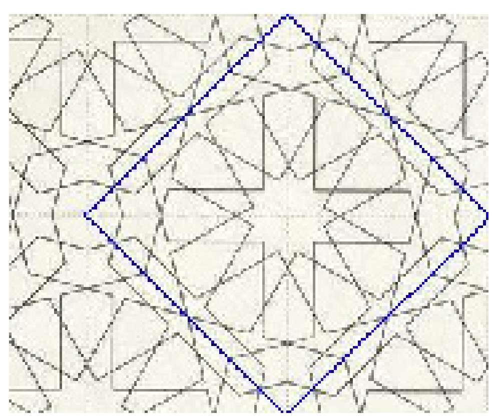


Figure 10b. *A portion of Bourgoin’s Plate 77, cropped by author, with a repeat unit outlined*

Bourgoin's Plate 118 (12- and 8-stars)

The third of the *p*4*m* patterns may also be found as a decorative wooden panel in the Masjid/Madrasa al-Sultan Barquq, dating to 1386. The 12-stars in this pattern may be achieved by first creating a {24/9} star figure and erasing half of the points (as shown previously in **Figure 9e**). This pattern also contains four, eight-pointed *regular* {8/3} star polygons surrounding each 12-star. To create an {8/2} star figure, connect with line segments every second of eight equally-spaced points on a circle (**Figure 11a**). After erasing some line segments, a more decorative 8-star is formed (**Figure 11b**). Similarly, to create an {8/3} star polygon, connect with line segments every third of eight equally-spaced points on a circle (**Figure 11c**). After erasing some line segments, a more decorative 8-star is formed (**Figure 11d**). A cropped version of Wade's catalog *EGY 1711* is shown in **Figure 11e**, along with a possible square repeat unit, shown in **Figure 11f**.

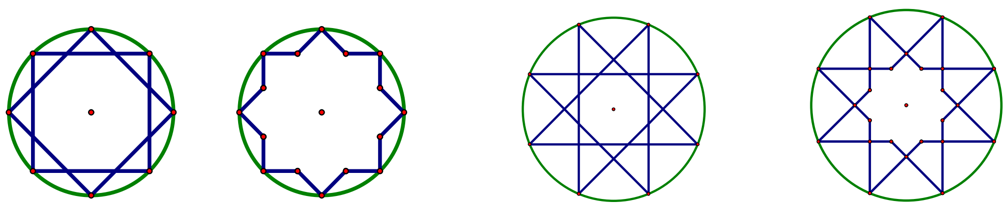


Figure 11a. *An {8/2} star figure*

Figure 11b. *A more decorative {8/2} star*

Figure 11c. *An {8/3} star polygon*

Figure 11d. *A more decorative {8/3} star*

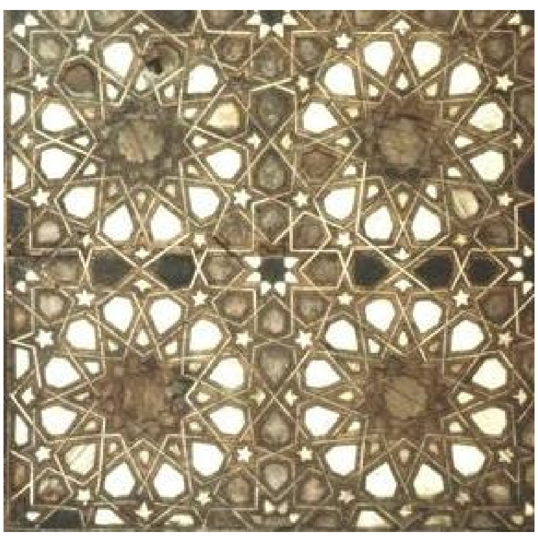


Figure 11e. *A decorative wood panel in the Masjid/Madrasa al-Sultan Barquq, EGY1711, dated 1386*

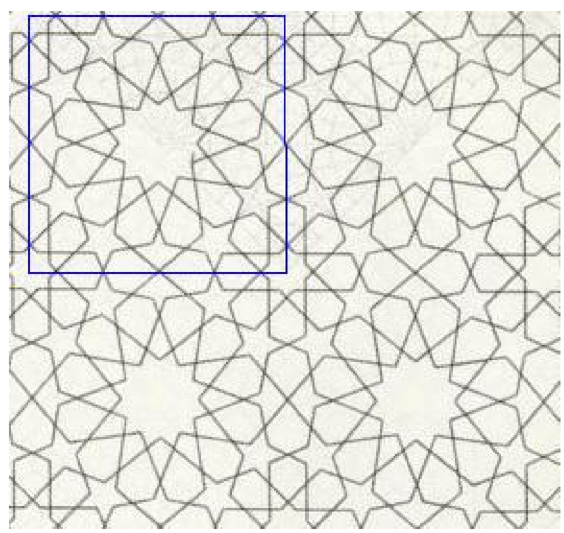
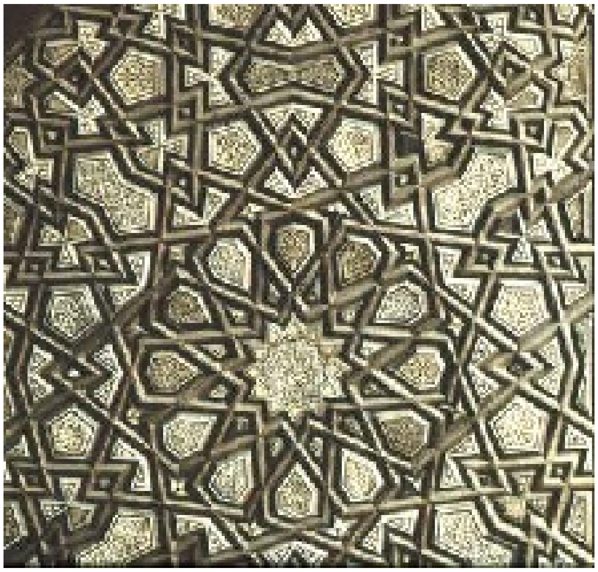


Figure 11f. *A portion of Bourgoin's Plate 118, cropped by author, with a repeat unit outlined*

Bourgoin's Plate 117 (12- and 8-stars)

The fourth of the *p*4*m* patterns may be found on yet another painted wood panel of a minbar in the Masjid al-Sultan Qaytbay, dating to 1474. The pattern consists of {12/5} star polygons and also {8/2} stars (see **Figure 11b**) – four surrounding each 12-star. A cropped version of Wade's catalog *EGY 1402* is shown in **Figure 12a** on the next page, along with a possible repeat unit, shown in **Figure 12b**.



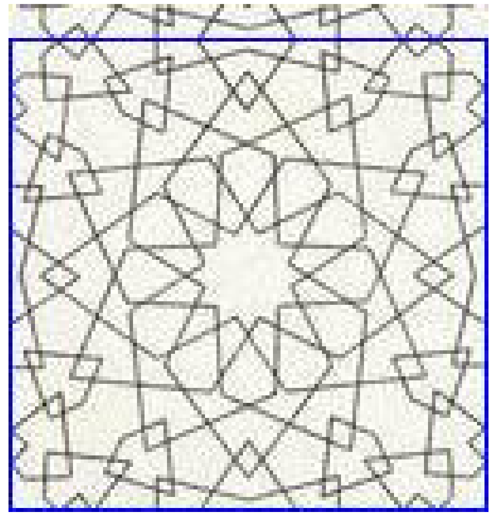
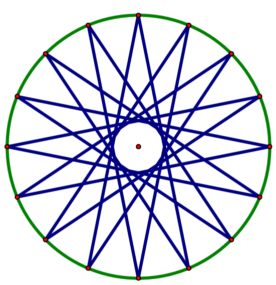


Figure 12a. *A wood panel on the minbar of the Masjid al-Sultan Qaytbay, EGY1402, dated 1474*

Figure 12b. *A portion of Bourgoin's Plate 117, cropped by author, with a repeat unit outlined*

Bourgoin's Plate 132 (16- and 12-stars)

The last three *p*4*m* designs have combinations of *regular* 16- and 12-stars. The first of these may be found on the main door to the Masjid al-Mu'ayyad Shaykh, dating to 1422. It is fabled that this copper door was pilfered from the Masjid/Madrasa of Sultan Hasan [9]. The pattern consists of large {16/7} star polygons surrounded by four {12/5} star polygons. To create a {16/7} star polygon, connect with line segments every seventh of sixteen equally-spaced points on a circle as shown in **Figure 13a**. After erasing some line segments, a more decorative 16-star is formed, as shown in **Figure 13b**.



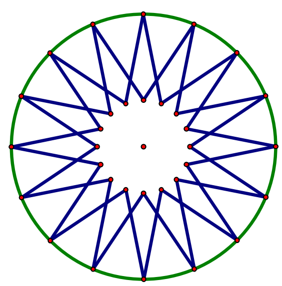
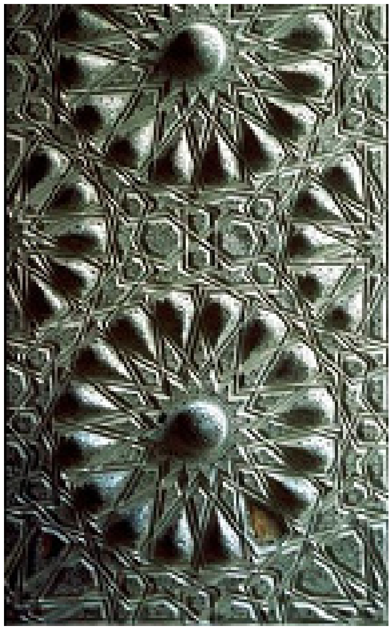


Figure 13a. *A {16/7} star polygon*

Figure 13b. *A more decorative {16/7} star*

A cropped version of Wade's catalog *EGY1222* is shown in **Figure 13c** on the following page, with a possible square repeat unit shown highlighted on a portion of Bourgoin's Plate 132 with the 16-star at the center (**Figure 13d**). Notice also that there are pairs of "nearly regular" heptagons in the interstitial space between the major stars and located at the midpoint of each edge of the square repeat unit. In the next design, 7-pointed stars will replace the heptagons in these locations.



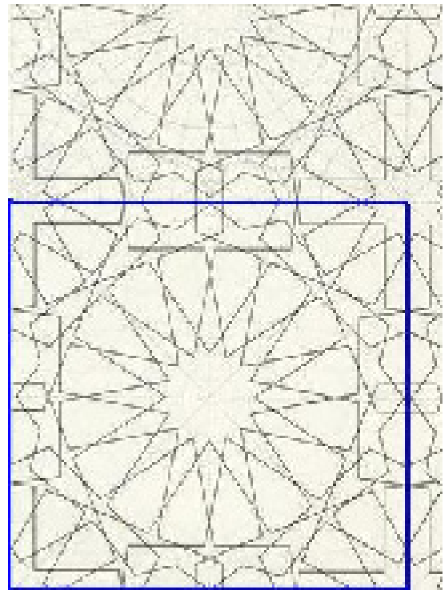
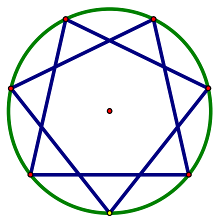


Figure 13c. *A copper door of the Masjid al-Mu'ayyad Shaykh, EGY,1222, dated 1422*

Figure 13d. *A portion of Bourgoin's Plate 132, cropped by author, with a repeat unit outlined*

Bourgoin's Plate 135 (16-, 12- and 7-stars)

The next *p*4*m* pattern is a highly unusual one because it consists of large {16/7} star polygons formed by arcs which are surrounded by twelve and eight stars formed from straight line segments. There are four {12/5} star polygons and eight smaller {7/2} star polygons. To create a {7/2} star polygon, connect with line segments every second of seven equally-spaced points on a circle as shown in **Figure 14a**. After erasing some line segments, a more decorative star is formed as shown in **Figure 14b**.



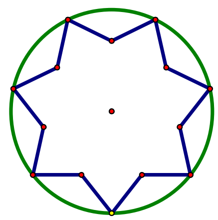


Figure 14a. *A {7/2} star polygon*

Figure 14b. *A more decorative {7/2} star*

The pattern may be found on a wooden panel inlaid with ivory and zarnashan (a precious metal inlay in objects made of another metal such as copper) [10] on a minbar in the Masjid al-Amir Qijamas al-Ishaqi, dating to 1481 (**Figure 14c** on the following page). A possible square repeat unit is shown highlighted on a portion of Bourgoin's Plates 135 (**Figure 14d**), with 16-stars at the center of each repeat unit. Notice also that pairs of 7-stars replace the heptagons that appeared in the interstitial space between the major stars in the previous pattern (Plate 132).

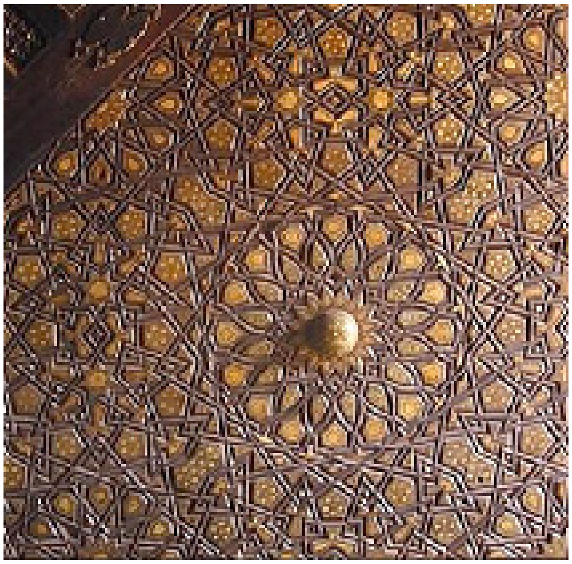


Figure 14c. *A wood panel on the minbar of the Masjid al-Amir Qijamas al-Ishaqi, dated 1481*

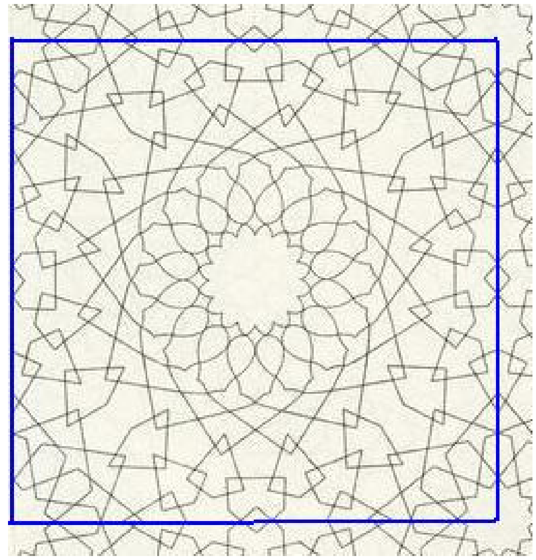


Figure 14d. *A portion of Bourgoin's Plate 135, cropped by author, with a repeat unit outlined*

Bourgoin's Plate 131 (16- and 12-stars)

The last of the *p*4*m* patterns may be found on a bronze door of the Madrasa al-Sultan al-Nasir Hasan, dating to 1363 (see **Figure 15a**). The pattern consists of four 12-stars surrounding large 16-stars all of which are formed by arcs, and thus, is completely devoid of line segments, as was the case for the last of the *p*6*m* star polygon pattern, Plate 121 (which also happens to be found on a bronze door to the Masjid al-Sultan al-Nasir Hasan; see **Figure 8a**). A possible square repeat unit is shown highlighted on a portion of Bourgoin's Plate 131 (**Figure 15b**), with the 16-star at the center.

Figure 15a. *A bronze door of the Madrasa al-Sultan al-Nasir Hasan, dated 1363*

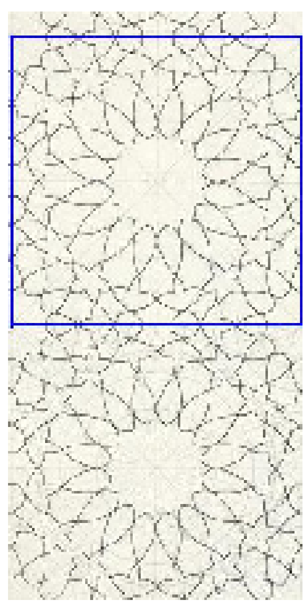


Figure 15b. *A portion of Bourgoin's Plate 131, cropped by author, with the repeat unit outlined*

Discussion

Fifteen of Bourgoin's twelve-star designs have been identified on historic monuments dating from the Mamluk period, with seven patterns coming from the forty year period of 1347 to 1386. All are highly symmetric, exhibiting either four-fold or six-fold rotational symmetry with multiple mirror reflections. Seven of the designs may be classified as belonging to the *p*4*m* crystallographic symmetry group, while the remaining eight may be classified as *p*6*m* patterns. Seven of the designs are comprised of only large twelve-pointed stars, while the other eight are twelve-stars in combination with *regular* (or *nearly-regular*) seven-, eight-, nine- or sixteen-pointed stars. Seven of the patterns are found as wood carvings, while those remaining are equally divided between being fashioned into bronze doors or worked into stone or mosaic. The {12/5} star polygon appeared most often (in 11 patterns) and two 12-stars were created by arcs alone, interestingly enough found at the same location, the Masjid/Madrasa al-Sultan al-Nasir Hasan Complex, dating to 1363.

As for finding any examples of extant Bourgoin's 12-star designs that may be generated from *Tashkent Scrolls'* repeat units, there is one for a 9- and 12-pointed star that does produce a design similar, but not identical, to Bourgoin's Plate 120. The Bourgoin plate contains {12/5} and {18/7} (with half of the points removed) stars and the Tashkent Scrolls repeat unit generates a pattern with {12/4} and {9/3} stars (**Figures 16a - f**). So, although the structures are the same, the shape and type of star polygons are different.

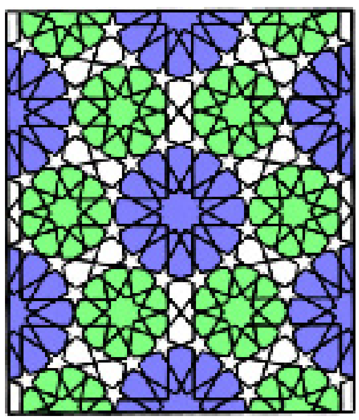


Figure 16a. *The author's reconstruction of Bourgoin's Plate 120*

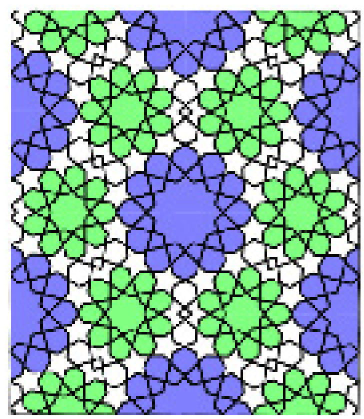


Figure 16b. *The author's reconstruction of the Tashkent Scrolls design*

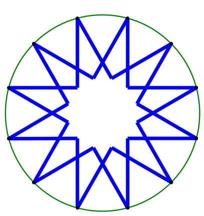


Figure 16c. *Bourgoin's {12/5} star design*

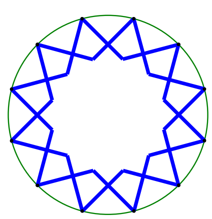


Figure 16d. *The Tashkent Scrolls' {12/4} star design*

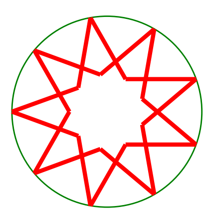


Figure 16e. *A Bourgoin's {18/7} star design*

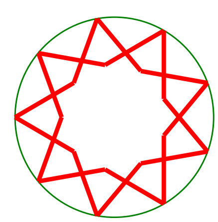


Figure 16f. *The Tashkent Scrolls' {9/3} star design*

The *Topkapı Scroll* has a total of only six repeat units that generate planar 12-star designs (thus excluding the sketches for muqarnas), and only three are close to matching the designs in Bourgoin's Plates 78, 118 and 126. Bourgoin's Plate 118 has {8/3} star polygons, while catalog number 35 of the *Topkapı Scroll* has {16/5} stars (with half of the points erased). Bourgoin Plates 78 and 126 differ from catalog numbers 63 and 44 (respectively) of the *Topkapı Scroll* in that both of Bourgoin's plates have three pentagonal stars surrounding a hexagon in place of the *Topkapı Scrolls'* use of a trilobed polygon, as shown in **Figures 17a** and **b**.

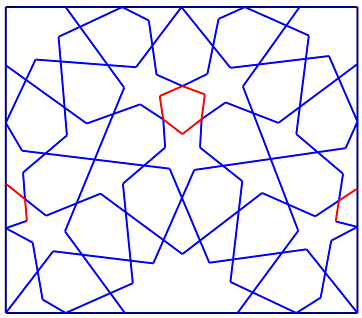


Figure 17a. *The author's reconstruction of Bourgoin's Plate 78*

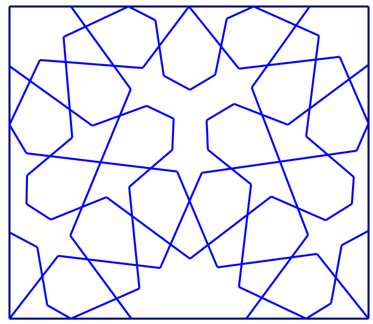


Figure 17b. *The author's reconstruction of the Topkapı Scrolls CN 63 repeat unit*

As the author was searching for matching patterns, many examples were close but not exactly the design shown in Bourgoin's plates and so those were disallowed. However, it does testify to the great variety possible in geometric Islamic patterns found in Cairo and elsewhere.

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Patterns on Triply Periodic Uniform Polyhedra

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Abstract

Artists have created patterns on closed polyhedra and on the hyperbolic plane, but no one to our knowledge has created patterns on triply periodic polyhedra. The goal of this paper is to exhibit a few such patterns and explain how they arose.

1. Introduction

In this paper we show patterns on triply periodic polyhedra, which are polyhedra that have translation symmetries in three independent directions in Euclidean 3-space. Figure 1 shows (a piece of) such a polyhedron decorated with angular fish and colored backbone lines. Each of the polyhedra we discuss is composed of

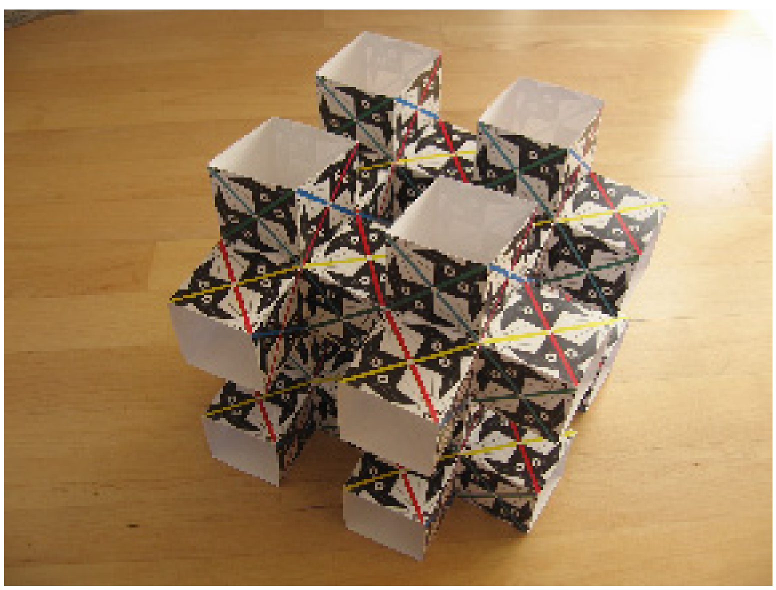

Figure 1: A hyperbolic Truchet tiling based on the {4,6} grid.

copies of a regular polygon, either a square or an equilateral triangle. These polyhedra have negative curvature, and are related to regular tessellations of the hyperbolic plane. Similarly, the patterns we place on the polyhedra are related to patterns of the hyperbolic plane based on regular tessellations.

This work was inspired by Luecking's polyhedral approximation of one of Sherk's minimal surfaces shown at ISAMA 2011 [Luecking], and by Chuang, Jin, and Wei's beaded approximation to Schwarz's D and Schoen's G surfaces shown at the Art Exhibit of the 2012 Joint Mathematics Meeting [Chuang]. It turns out that the polyhedra we discuss are related to triply periodic minimal surfaces.

The Dutch artist M.C. Escher drew patterns on several closed polyhedra [Schattschneider04]. Later Doris Schattschneider and Wallace Walker designed non-convex rings of polyhedra, called Kaleidocycles, which could be rotated [Schattschneider05]. The purpose of this paper is to investigate patterns on polyhedra that could theoretically extend to infinity in all three directions. Thus we were naturally led to using triply periodic polyhedra.

We will begin with a discussion of regular tessellations and triply periodic polyhedra, explaining how they are related via minimal surfaces. This relationship also extends to patterns on the respective surfaces. Then we show two patterns on what is probably the simplest triply periodic polyhedron, followed by a pattern on a more complicated polyhedron. Finally, we indicate possibilities for other patterns on other surfaces.

2. Regular Tessellations and Triply Periodic Polyhedra

We use the Schläfli symbol $\{p, q\}$ to denote the regular tessellation formed by regular p-sided polygons or *p-gon* with q of them meeting at each vertex. If $(p-2)(q-2) > 4$, $\{p, q\}$ is a tessellation of the hyperbolic plane, otherwise it is Euclidean or spherical. Figure 2 shows the tessellation $\{4, 6\}$ in the Poincaré disk model of hyperbolic geometry. Figure 3 shows a pattern of angular fish based on that tessellation.

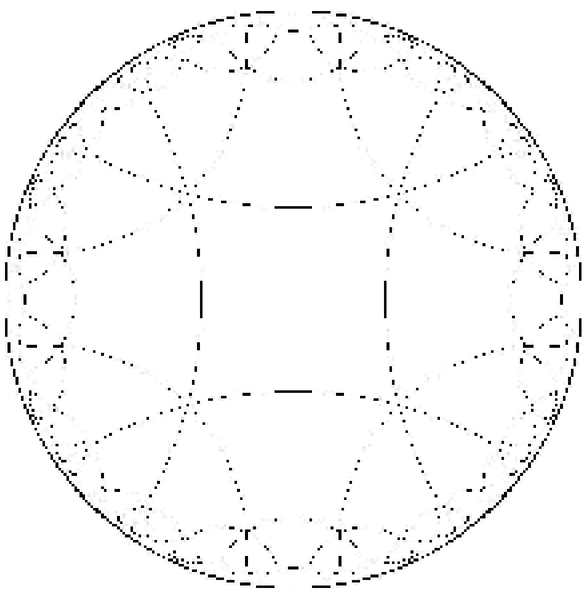

Figure 2: The $\{4,6\}$ tessellation

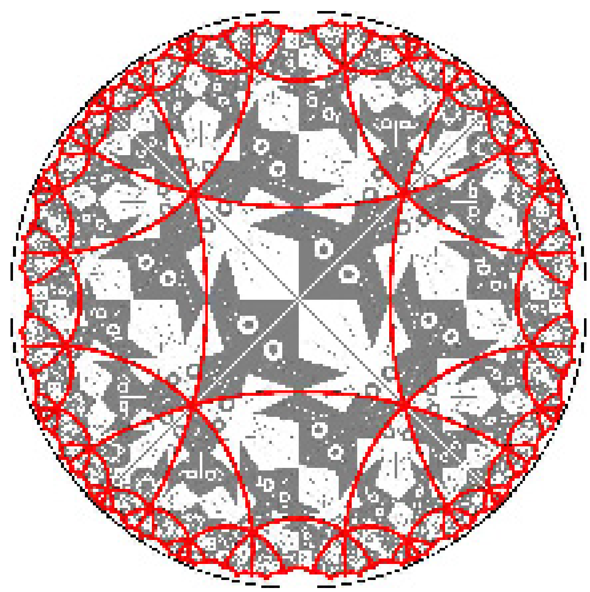

Figure 3: The $\{4, 6\}$ superimposed on a pattern of fish.

We will discuss *regular triply periodic polyhedra* which have a p-gon for each of its faces, has translation symmetries in three independent directions, and its symmetry group is transitive on vertices (i.e. it is uniform) at which q p-gons meet. The meaning of the Schläfli symbol $\{p, q\}$ can be extended to include these polyhedra. If the symmetry group is flag-transitive, then one obtains a more symmetric subfamily called *infinite skew polyhedron.* There are exactly three of them, which was discovered by John Petrie in 1926 [Wiki]. H.S.M. Coxeter designated them by the extended Schläfli $\{p, q|n\}$, indicating that there are q p-gons around each vertex and n-gonal holes [Coxeter73, Coxeter99]. Figure 1 shows $\{4, 6|4\}$ with a pattern of fish on it. The other possibilities are $\{6, 4|4\}$ and $\{6, 6|3\}$.

There is a 2-step connection between some regular triply periodic polyhedra $\{p, q\}$ and the corresponding regular tessellation $\{p, q\}$. First, some of the periodic polyhedra are approximations for (and closely related to) triply periodic minimal surfaces (TPMS). Alan Schoen has done extensive investigations into TPMS [Schoen]. Second, each smooth surface has a *universal covering surface*: a simply connected surface with a covering map onto the original surface which is a sphere, the Euclidean plane, or the hyperbolic plane. Since each TPMS has negative curvature, its universal covering surface is the hyperbolic plane. Similarly, we might call a hyperbolic pattern based on the tessellation $\{p, q\}$ the "universal covering pattern" for the related pattern on the polyhedron $\{p, q\}$.

3. A Pattern of Angels and Devils on the $\{4, 6|4\}$ Polyhedron

One can see by examining Figure 1 or Figure 5 below that the $\{4, 6|4\}$ polyhedron is based on the cubic lattice in 3-space. In fact it divides 3-space into two complementary congruent solids. Each of the solids is composed of "hub" cubes with "strut" cubes on each of its faces; each strut cube connects two hub cubes. M.C. Escher's hyperbolic print *Circle Limit IV* of angels and devils was based on the $\{6, 4\}$ tessellation. Figure 4 shows an angels and devils pattern based on the $\{4, 6\}$ tessellation. Figure 5 shows the corresponding pattern on the $\{4, 6|4\}$ polyhedron.

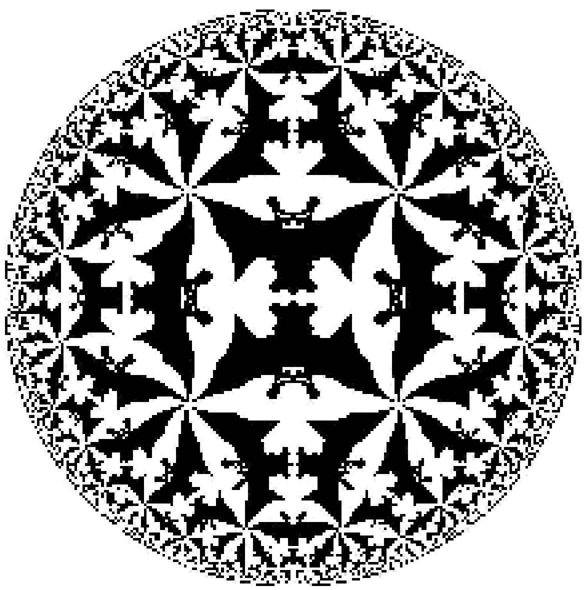

Figure 4: Angels and Devils on the $\{4, 6\}$ tessellation.

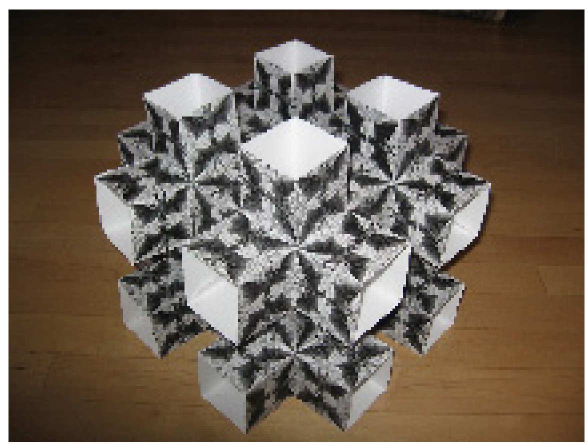

Figure 5: Angels and Devils on the $\{4, 6|4\}$ polyhedron.

The lines of bilateral symmetry of the angels and devils on the polyhedron form squares around the "waists" of the struts and "strut holes". If these squares are relaxed to circles (touching at the feet of the angels and devils), and the space between the circles is spanned by a minimal surface, one obtains Schwarz's P-surface, a TPMS. We will look at this more closely in the next section.

4. A Pattern of Fish on the $\{4, 6|4\}$ Polyhedron

Figure 2 above shows a pattern of angular fish in the hyperbolic plane — in fact it is the "universal covering pattern" of the fish on the $\{4, 6|4\}$ polyhedron shown in Figure 1. This pattern of fish is related to Escher's hyperbolic pattern *Circle Limit I* in the same way that Figure 5 is related to Escher's *Circle Limit IV* — Figures 2 and 5 are both based on the $\{4, 6\}$ tessellation, whereas Escher's hyperbolic patterns are both based on the $\{6, 4\}$ tessellation.

In the pattern of Figure 1, the backbones of the fish in a horizontal plane lie along parallel red lines or parallel yellow lines that are perpendicular to the red lines. Similarly for planes facing the lower left, fish backbones lie along green and cyan lines; and for planes facing the lower right, the backbones lie along blue and magenta lines. Also there are two kinds of vertices: those where red, blue, and green backbone lines intersect (at 60 degree angles), and those where cyan, magenta, and yellow lines intersect. Figure 6 shows a close-up view of the latter kind of vertex. Figure 7 shows the universal covering pattern of Figure 1, including the colored backbone lines. The backbone lines of the fish in Figures 1 and 6 are closely related to Schwarz's P-surface, since they all lie on that surface. Figure 8 shows the P-surface with the embedded lines [IndianaP]. One can see that they are the same as the backbone lines of Figure 1.

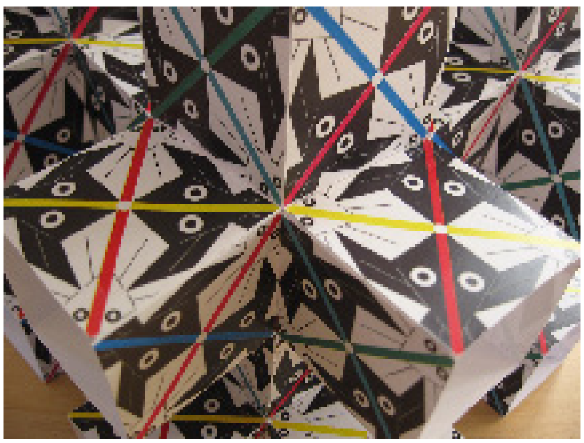

Figure 6: A vertex view of the fish pattern on the $\{4, 6|4\}$ polyhedron.

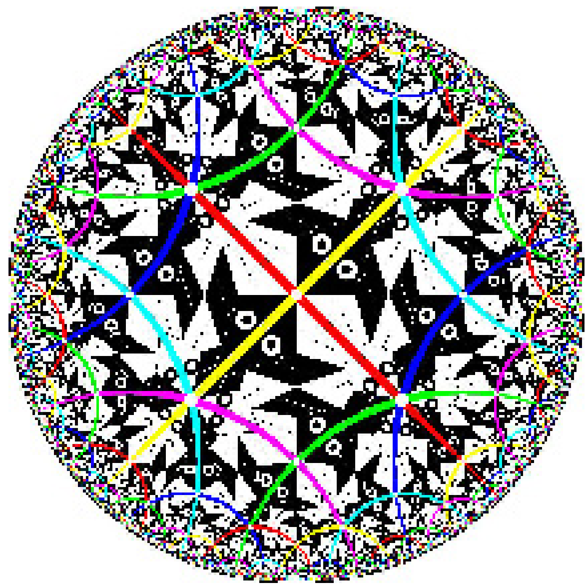

Figure 7: A hyperbolic pattern corresponding to Figure 1.

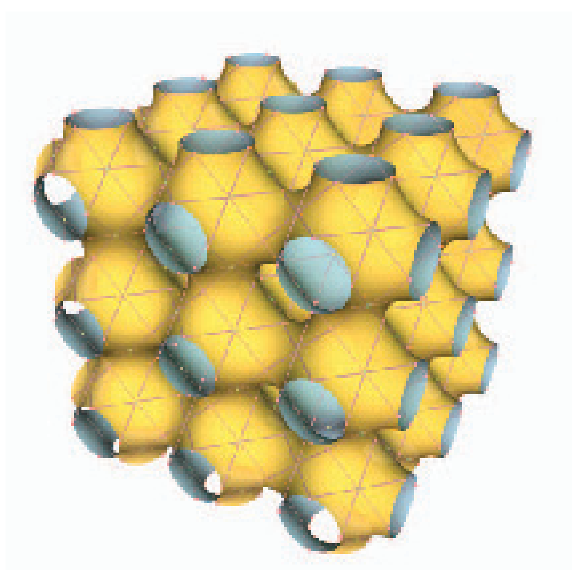

Figure 8: Schwarz's P-surface showing embedded lines.

5. A Pattern of Fish on a $\{3, 8\}$ Polyhedron

Figure 9 shows a triply periodic $\{3, 8\}$ polyhedron (Figure 5(2) of [Hyde]). The symmetry group is not flag-transitive, since there are two kinds of equilateral triangles. This surface can also be described in terms of hubs and struts, both of which are regular octahedra. A hub octahedron has strut octahedra on alternate faces, so that four hub triangles are covered by struts and four remain exposed. Each strut octahedron connects two hubs, and thus has two of its faces covered by hubs. The exposed hub triangles are different from the exposed strut triangles. Figure 10 shows a pattern of fish on the $\{3, 8\}$ polyhedron.

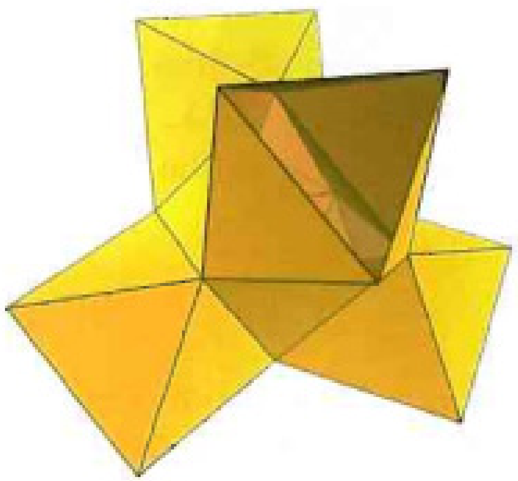

Figure 9: A triply periodic $\{3, 8\}$ polyhedron.

Figure 10: A pattern of fish on the $\{3, 8\}$ polyhedron.

The fish were inspired by those in Escher's *Circle Limit III* print. Figure 11 shows the corresponding $\{3, 8\}$ tessellation superimposed on a computer generated rendition of *Circle Limit III*.

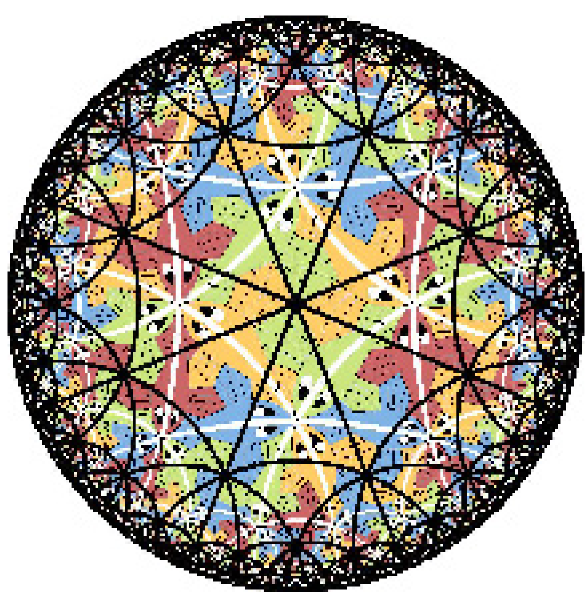

Figure 11: A rendition of *Circle Limit III* with the $\{3, 8\}$ tessellation superimposed.

The $\{3, 8\}$ polyhedron is an approximation to Schwarz's D-surface, a TPMS which also has embedded lines. The "D" is used to denote this surface since it has the shape of a thickened diamond lattice. The red, green, and yellow fish of Figure 9 swim along polygonal approximations to those embedded lines. The blue fish swim around hexagons which form the "waists" of the strut octahedra.

More than 30 years ago I tried piecing together small paper octagons with colored lines on them corresponding to the lines of fish in *Circle Limit III*. I was naturally led to a diamond lattice structure, and thought it would work, but was not sure until I undertook this project.

6. Observations and Future Work

We have shown some patterns on two triply periodic polyhedra. In fact we showed two different patterns on the $\{4, 6|4\}$ polyhedron. One of our goals was to find patterns that would emphasize interesting mathematical properties of the polyhedra — having linear elements that correspond closely to embedded lines in related minimal surfaces, for instance.

Although it has been known for 85 years that there are only three infinite skew polyhedron, the more general uniform triply periodic polyhedra have not been classified, but a number of examples are known. It would be challenging not only to place mathematically and artistically interesting patterns on the known polyhedra, but even more challenging to discover new polyhedra.

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Variations on 45 Degrees and Cutting and Stacking

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Abstract

Sculptures are discussed where components are set at angles of +,– 45 degrees, as well as sculptures where I-beams or tubes are cut at 45 degrees and stacked.

Variations on 45 degrees

Variation I is shown in Figure 1(a) and (b). Note that there are three small isosceles right triangles formed by three sections of angle iron that have angles of 45 degrees with the horizontal. This results in the lower and upper long rectangular bars being at an angle of 45 degrees and the rear lower long rectangular bar is at - 45 degrees. The center bar is horizontal. The edges of the square are +, - 45 degrees. There are also spaces between components with boundary lines of +, - 45 degrees and 0 degrees.

(a) (b)

Figure 1. (a) Variation I, Steel, 2011. (b) Detail.

Variation II is shown in Figure 2. Comments in the case of Variation I apply to Variation II. The square in Variation I has been replaced by a section of square tube in Variation II. Angle iron sections were used for the long rectangular bars in Variation I. I-beam sections were used for the long rectangular bars in Variation II resulting in additional shadows. These configurations were influenced by certain stainless steel cubi sculptures of the pioneer American sculptor David Smith.

(a) (b)

Figure 2. (a) Variation II, Steel, 2011, (b) Detail.

Cutting and Stacking

Steel sculptures constructed by cutting and stacking were introduced in [1]. Here we discuss some additional examples. The first is Sculpture 3 in Figure 3(a) consisting of two circular tubular sections cut at 45 degrees.

(a) **(b)**

Figure 3(a) Sculpture 3, Steel, 2012, (b) Sculpture 4, Steel, 2012.

The second is Sculpture 4 consisting of rectangular tube end cuts at 45 degrees where the cut was made so that the top view is an isosceles right triangle allowing the two components to be stacked with an inversion and rotation. The very thin linear edge is similar to the very thin part of the curved edge in Sculpture 3. Note that both sculptures 3 and 4 are double tori, where the spaces (holes) are the prominent features. The thin edges cast dramatic shadows against the opposite side when the sun is low in the right direction. A thin shadow can be seen in the lower part of Figure 3(b).

Forms in Space: Family Groups

In Figure 4(a) we have a stacking of three 45 degree end cuts with an I-beam section supporting a rectangular tube section containing an I-beam section. The 45 degree cut allows for a larger opening as well as a view into the space.

(a) **(b)**

Figure 4. (a) Family Group I , Steel, 2012, (b) Family Group II, Steel, 2012.

In Figure 4(b) we have a stacking of four 45 degree end cuts with an I-beam section supporting a rectangular tube section containing an I-beam section supporting a rectangular tube section. If the lower I-beam section is considered as the supporting father and the enclosing tube section is regarded as the mother, then we can consider the sculptures as family groups as indicated in the titles. Note that Family Group II can be considered as a two-stage fractal.

We emphasize that the 45 degree end cuts in Figure 4 are better than using regular rectangular cuts because it is easier to see into the end spaces with 45 degree cuts. Note that this is also the case in Figure 3.

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Contemporary Tilings

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Abstract
By presenting several examples, this article aims to persuade those interested in tiling to take advantage of the new possibilities provided by the new technology of computers to introduce contemporary tilings into the world of architecture.

Introduction

As one of the most artistic kinds of ornaments, tiling has historically been interwoven with architecture. By incorporating simple or complex configurations of Euclidean geometry with their artistic tastes for colors, traditional artists and craftsmen created beautiful masterpieces. However, developing new innovative configurations, inevitably, entails time-consuming practices in form of trial and error. Geometry is a strict science and aesthetics only appears through the correctness and strictness of configurations based on geometric rules that introduce complexity. But, nowadays, with the aid of computers, algorithms, and coded mathematics many tiring and time-consuming works are done by means of computers. They can carry out thousands of calculations and comparisons in the blink of an eye and artists can therefore effectively experiment with innovations.

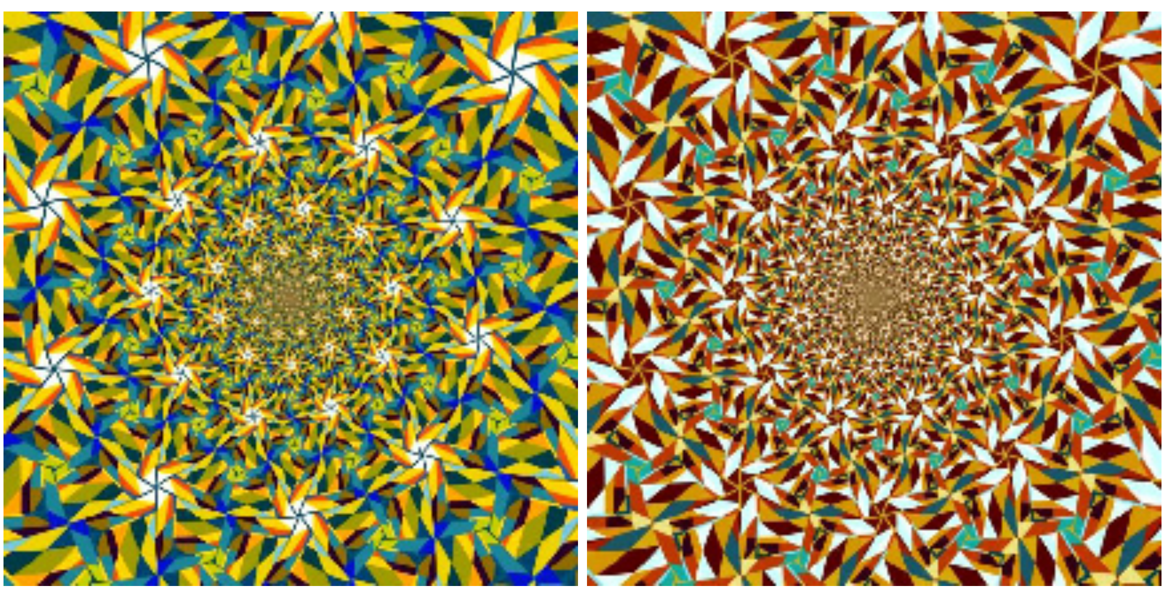


Figure 1

We have digitally created eight pairs of tiling sets below, each of which is the result of experimenting with hundreds of possibilities for the parameters defining forms and colors. All of them are created in Ultra Fractal [1], a professional fractal making program, and they are central tilings which are of interest to the authors as a new type of modern tiling that tends towards infinity at the centre. The geometries of different pairs are, more or less, different but still related to each other structurally.

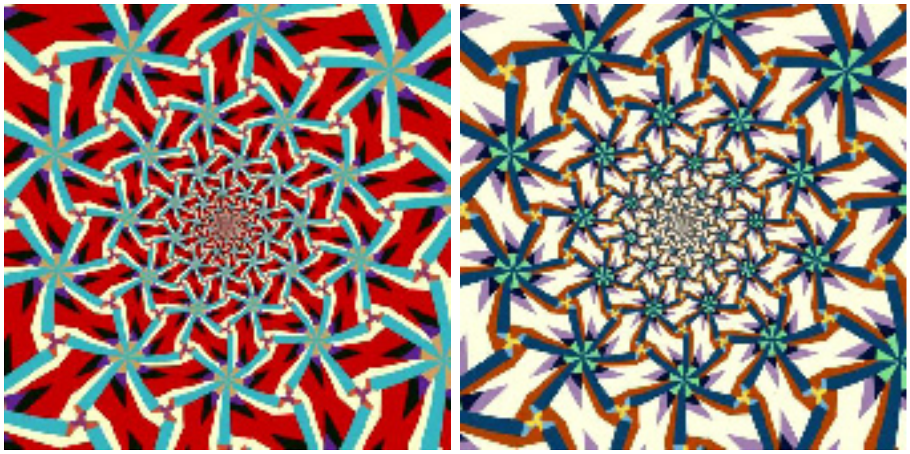


Figure 2

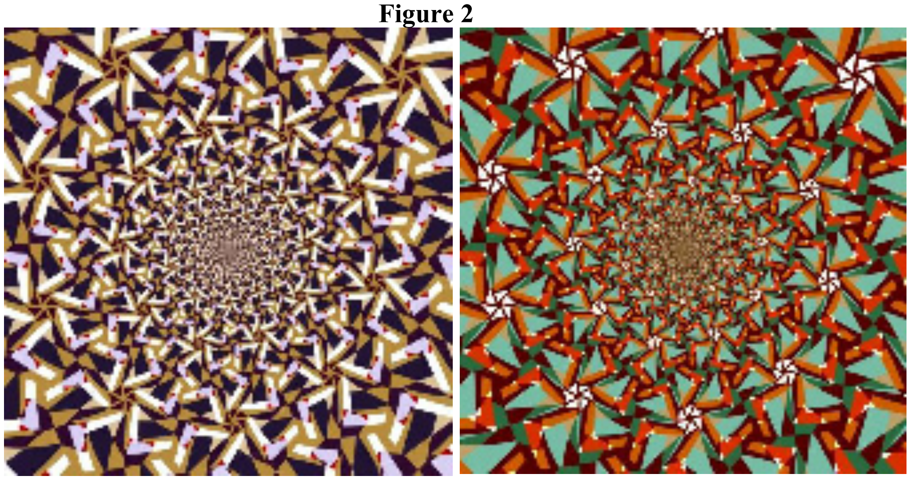


Figure 3

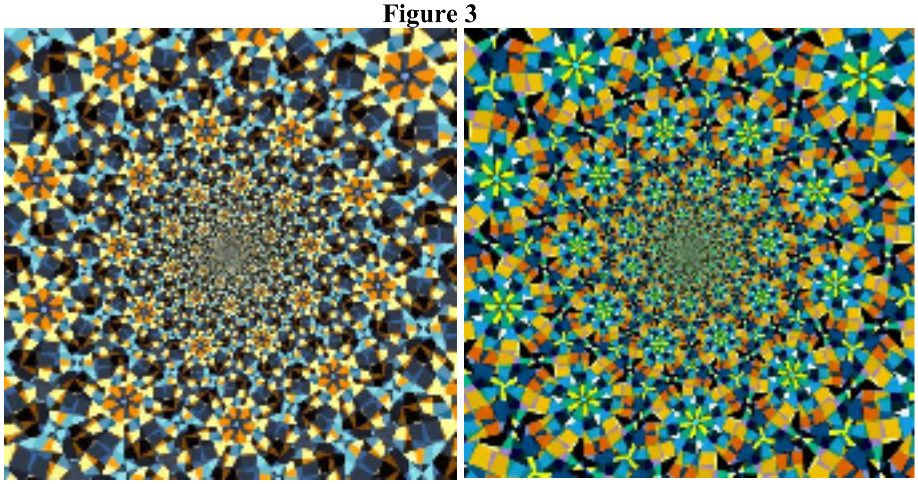


Figure 4

Each pair of tilings in a row has the same pattern but different coloring methods, to show how digital media can help to have entirely different visual results just by the possibility of experimenting with different color adjustments. All pairs require trying a lot of defining parameters and each pair requires trying a lot of color gradients. One of the benefits of digital media here was that, for example, we could change the color of all the tiles of the same color with a simple click at once, and even could change all the colors of the entire tiling by moving the controlling bar of the overall gradient. So, we had an infinity of choices available in front on the screen and could compare them easily to select the best. If we had the traditional instruments of paper, ruler, and watercolor it might take us months to come up with every pair.

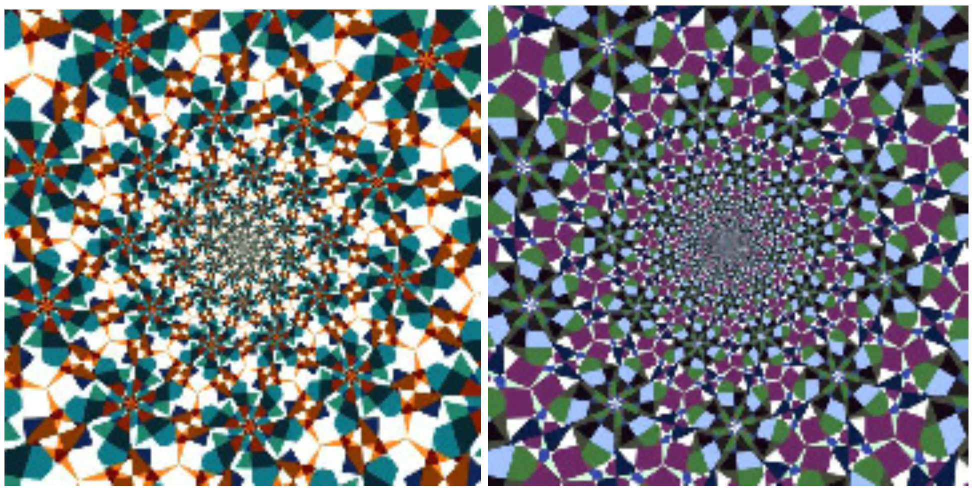
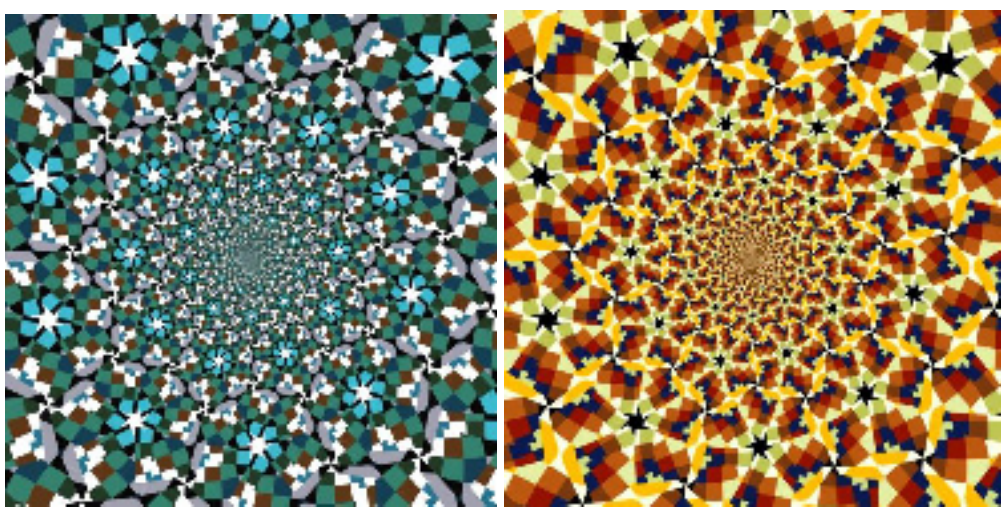


Figure 5



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SculptGen and Animation

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Abstract
Carlo Sequin's SculptGen is combined with animation to embellish the creation of closed toroidal sculptures.

In every artistic process, the artist creates works based on an initial model that he/she has in mind and also according to the material and techniques that are available. The result is also determined by the artist's aesthetical intuition and the limitations over materials and tools. In modern mathematical art, the basic tools and materials which are available to artist are usually embedded in programs that make it possible to create and present shapes and models digitally. Some of these software packages are fundamentally designed for pure mathematical purposes and may feature artistic aspects only after a creative mind discovers their potential for introducing aesthetics and creativity into the world of numbers and equations.

Another group of programs are designed with an eye on the artistic side or are basically provided for artistic creation, such as *TopMod* [1], *SculptGen* [2], *Surfer* [3], *Ultra Fractal* [4], *Mandelbulb3D* [5], etc. Nonetheless, even these programs do not provide users with any ready artistic results at first click. They only provide artists and users with useful interfaces and tools to handle and settle things more easily and straightforwardly.

Theoretically most of the works of art-math programs could be reconstructed in more common and commercial 3D products like *Autodesk 3ds Max* [6]. However, even their reconstruction might take enormous time. However, it might take an exclusively designed math-art program just a few appropriate techniques to reach the final result. This way of availability of a collection of certain tools and functions in defined relationships plays the key role in bearing fruitful strategies and techniques during interaction with interface.

Gaining artistic results in such programs can be either very difficult or very simple but tricky! Easy solutions only need an open mind with enough skill over the software.
SculptGen is one of the math-art pieces of software which creates eye-catching topological sculptures out of minimal surfaces at a very high speed. This handy program, however, has only a small number of controllers and might sound very limited to beginners. Actually, it is the creative artist who can a lot of possibilities from very simple ideas. In the case of *SculptGen,* which is designed for still shapes, playing with handy controllers, I thought how nice it would be if it had the possibility of animating the transformation of forms.

Here, I describe a few simple strides that I took to attain short looped animations like *Self-Conversion* (Figure 1), playable at [7], and *Seven Pointed Star* (Figure 2), playable at [8], which were presented at *SIAF Symposium 2010* [9] and *GA2010* [10], respectively.

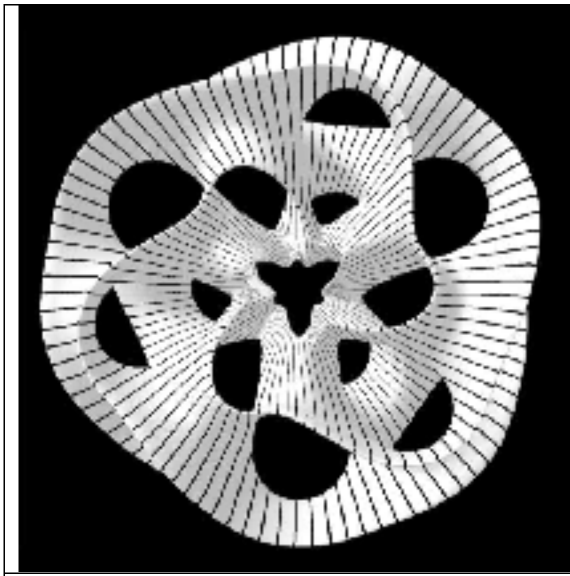

Figure 1: A still frame of the animation *Self-Conversion (2010, Mehrdad Garousi).*

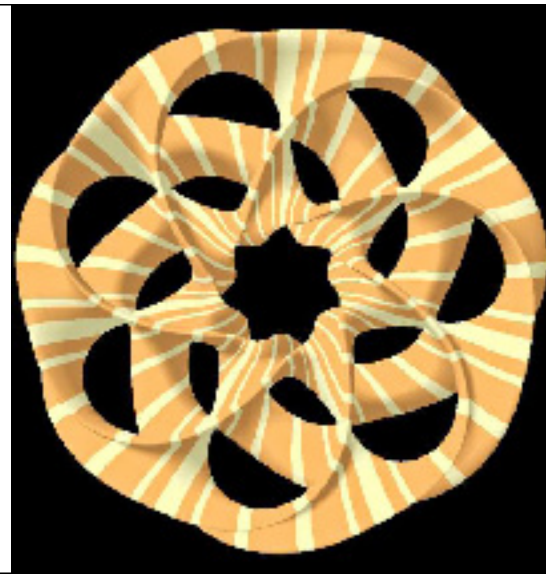

Figure 2: A still frame of the GIF animation *Seven Pointed Star (201, Mehrdad Garousi).*

We begin by creating a closed toroid by selecting proper amounts for "*Bend towards arch*" in accordance with other settings at "*#of saddle branches*", "*# of holes/storeys*", and "*Overall axial twist*" (Figure 3). Now that you have a closed toroidal shape, the key is that, you can twist it around its central axis by continuous changes in "*Turn around axis*". The degree intervals at which the toroid presents a complete loop depend on other settings. Now, you should record all changes of the toroid during its twist in a complete loop.

An easy and a bit painful solution is to move the "*Turn around axis*" controller unit by unit (in the software each unit is 5 degrees), press the "Prt Sc" (Print Screen) key on the computer keyboard, paste the captured screens one by one in new windows in an image processing program, and save them as single frames on hard drive. The next step will be cutting the unnecessary boarders of all frames, which takes totally a few clicks in professional image processing programs like *Adobe Photoshop* [11].

The final step is just joining existing frames by means of any frame-to-animation software and producing a smoothly playable animation with infinite loops. Now, if you play the animation, depending on the toroid used, due to the applied holes and twists in the body of the shape, you will have outstandingly complex morphing. If you put aside the simple mathematical rule behind the twist of the toroidal shape, and watch the movements from another point of view, you can find very accurately harmonic interchanges between positive (the sculpture) and negative (holes) spaces. Applying textures, preexisting in *SculptGen*, will also help to increase the reality, beauty, and complexity of the play.

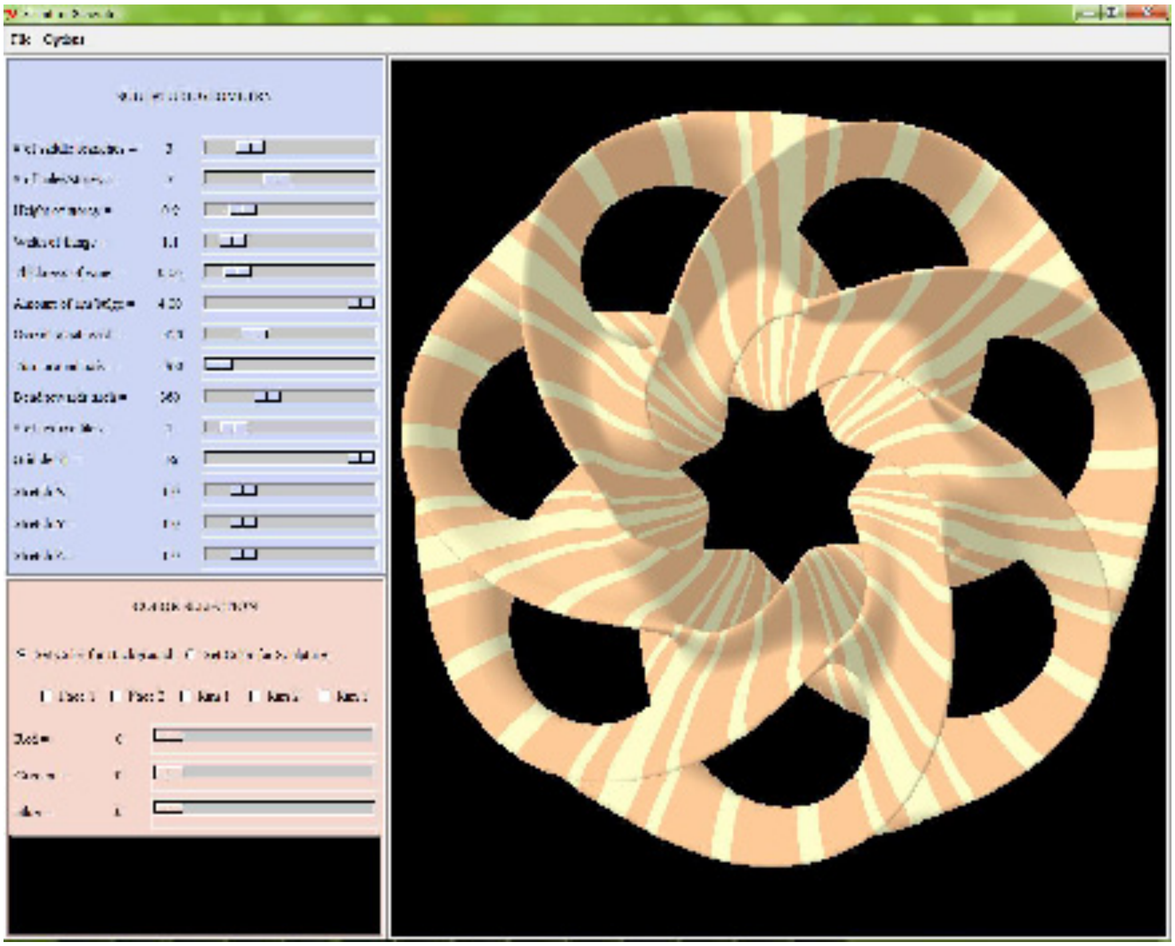

Figure 3: The interface of *SculptGen* during the creation of "Seven Pointed Star". By setting the controllers of your program the same as it could be seen above, you can have the exact shape in your viewer to start with and create your own shapes and animations.

A very useful tutorial about *SculptGen* entitled *Carlo Sequin's Sculpture Generator 1* could be found at [12] and the program is available free to download at [2].

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Seamless Night: Dream Time as Creative Inspiration

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An artist has the opportunity to supplement the scope of experience by amassing the imagery, symbolism and circumstances dramatized in dreams.

In 1997 I compiled data on how artists use dreaming as inspiration for art making. I interviewed twenty artists whose work varied greatly in medium. Each one tapped into dream memory for conceptualization, imagery, problem solving, and refinement. Many of the artists' works mirrored their experiences and emotional proclivity, (daily influences, like in dreams of social interaction or angst, as in this artist's sculpture in Fig. 2); while other artists' works represented their particular passions (like in dreams of vintage planes or flying). I then introduced these methodologies to enhance my strategies while teaching art classes, as well as in my own creative ventures.

Fig 1. Last Tango, crocheted wire, 2012 - characters in dreams have color and movement which influenced this sculpture. The form emerged from rearranged fragments. **Fig 2.** Grasp,crocheted wire- reflects emotional reaction to letting go of family symbolized in dreams

In the journey I confirmed that artists of protean abilities effectuated levels of dream time resource in the creative process. Additionally, fledgling artists (ages 7-9), while still in the nonconforming developmental stage, were eager and motivated to share their dream episodes, incorporating them pictorially.

The resulting enthusiasm had promising implications. Relevant to content, "every segment of a dream represents aspects of yourself"(LaBerge, p. 90). Wakeful experience is generated from the same mind as dream experience. For example, excerpts from a memorable vintage movie became the imagery content for "dream dancing" in this artist's sculpture featured in Fig. 1.

In that context "is it possible that the rules of the dream world could also apply to the physical world?" (LaBerge, p.91). Both dreams and reality arise, indeed, from the same memory storing brain. Therefore, we seem to have the option to change any aspect of self if we change the mind. Repeatedly incorporating dreams can lead to spontaneity: a possibly seamless transferal from sleep to awakening, harvesting from those frontiers that reciprocate reality and its unfettered array of components.

Research results published from 1974 through 1987 suggest that many of the techniques to exploit sleep time can be learned (Lish, 1997). While immersed in my study, from 1994 through 1997, I wrote in a dream journal, focused on problems about which I intended to dream; cleared my mind as much as possible, of noncreative activities; directed daytime thoughts to sculptural ideas, and set about reviewing these daytime happenings in the fabric of the night.

  

Fig 3. Crinolines-Dreaming solved a problem pertaining to movement I see in hyperbolic ruffles.
Fig 4. Festive Apparitions,crocheted dye ribbon,2012-my dream beings appeared faceless but nonthreatening. **Fig 5.** Contrasts,crocheted dye ribbon-alien to memory, shapes appear in dreams that suggest new arrangement.

While my ventures did yield increments of success, the complexities and fragments of my days, alien to my artistic counterpart, impeded efficiency: working full time, parenting, attending grad school, and working through splintered scraps of studio time.
Each moment lived is foundation for that which follows, in any dimension. In my eight years engaged in course work and research, that continuity provided groundwork for conceptual and concrete refinement.

My dream work, revisited, is a scramble still of frantic days and weeks, fragments of art time I can realistically pursue. The work I do is crocheted, knitted and beaded: one stitch at a time, one bead at a time, imbued with tactility- a guilty pleasure, accompanied by that rhythm of merging thought. Each artifact is envisioned, a transitional apparition that often was formalized, as in Fig. 3, Fig. 4, and Fig. 5. The culminating structure always differs from the originating plan, whether solidified in my dream or physical world.

In this creative process, dreams fit uninterrupted- a continual current of psychological and emotional responses. I can fuse image and expectation; plan and transformation.
I have witnessed so many contrasting realms: the undercurrent of thought and imagery. Insistent immersion in art is the constant that melds waking consciousness and the alternate reality of sleep.

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Plato's Blocks: Nested Spherical Polyhedrons from Modules

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The sculptor creates spherical polyhedrons whose interiors are hollowed to form an alternate polyhedron nested within the larger external polyhedron. The polyhedrons are all variations on Platonic, Archimedean, Catalan and Kepler-Poinsot solids. Sculptures comprise the repetition of a single, block-like module whose simple geometry belies the complexity of the finished work. Environmental versions of this series use blocks that may serve as seating and contain hollows in which children may nestle

Introduction

Sculptures shaped as spherical polyhedrons lend themselves to straightforward modular construction, since each face is the repetition of other faces. The construction process becomes considerably more intriguing when the modules in question combine to simultaneous form two different polyhedrons: one defined by the exterior of the sculpture and the other by a hollow in the interior of the sculpture. Splitting the sculpture into two parts reveals both the internal and the external polyhedrons of the sculpture. The effect is akin to opening a geode and discovering the crystalline formation at its center. Further, the inclusion of a distinct negative, played against an equally distinct positive provides effective sculptural contrasts.

In order to be effective, module design must fulfill three criteria. One set of faces must describe the exterior polyhedron, while a second set must describe the interior polyhedron. Finally, to ensure structural integrity, the side faces of the module must mate with the neighboring copies of the module.

Cubic Symmetry

For these sculptures modular construction works if the polyhedrons are members of the same symmetry group, either cubic or pentagonal. Cubic symmetries ensue when octahedral, hexahedral, tetrahedral or derivatives formations among the Archimedean and Catalan solids encase hollows of similar formation. The sculpture Boxed Hold (Figure 1), for example, features a cube with a hollow rhombic dodecahedron at its core.

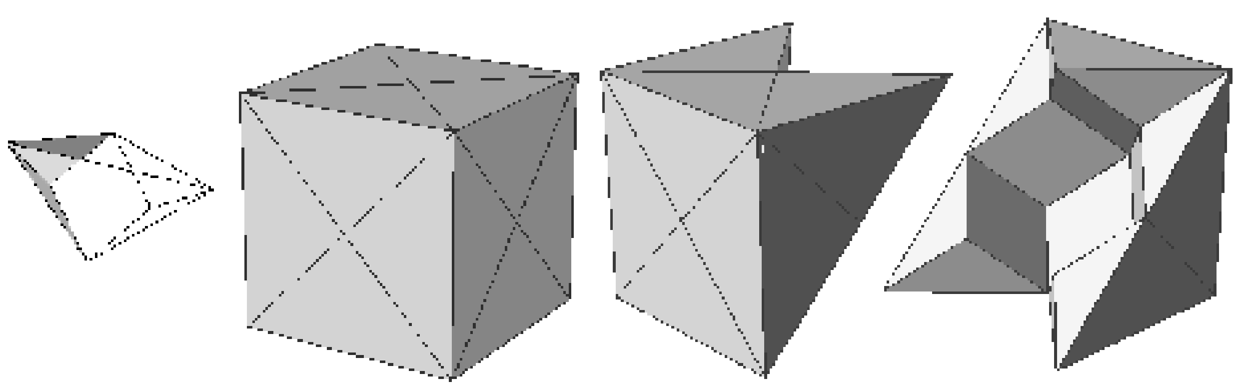


Figure 1. *Boxed Hold, Cube/Rhombic Dodecahedron. Shown with modular unit (left) and split open (right).*

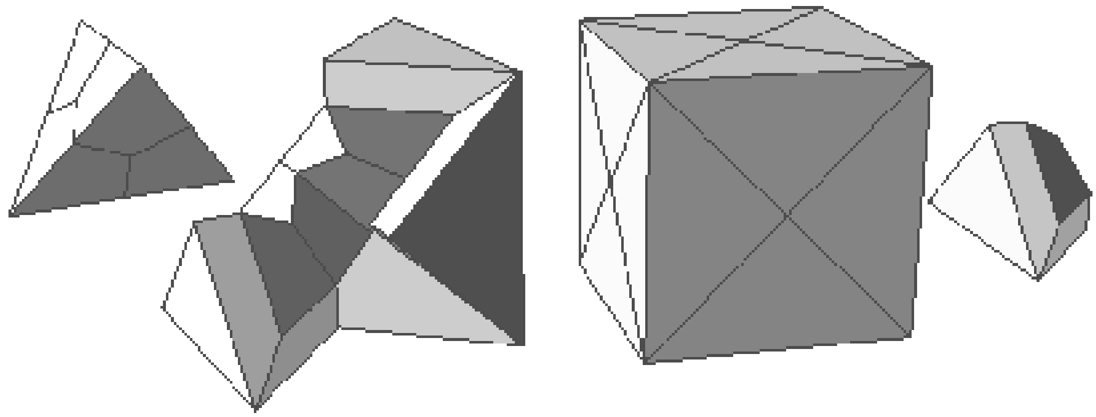


Figure 2. *Boxed Pyramid, cube/rhombic dodecahedron. Shown with a tetrahedron fitted to the interior (left), sculpture open (center left), sculpture closed (center right) and the modular unit (right).*

In Figure 1, the dodecahedron exists as a hidden hollow within the cube, as does the tetrahedron in Figure 2. The scale of the sculptures for exterior exhibit is such that this hollow may hold a crouching person, i.e., with an overall diameter of 66”. Gallery displays work best when the units fit comfortably in the viewer’s hand. At both scales public interaction is likely.

In order to reveal its interior the sculpture must display as two or more parts. This display may vary from location to location as permitted by rearranging the modules.

Since the 12 faces of the rhombic dodecahedron correspond to the 12 edges of the cube, both bear the same underlying symmetry. Similarly the six edges of the tetrahedron both may correspond to the six faces of the cube, thereby also fixing these into the same symmetry group. Figure 3 capitalizes on this by inverting the relationship in Figure 2 and situating a cube as the nucleus of the tetrahedron.

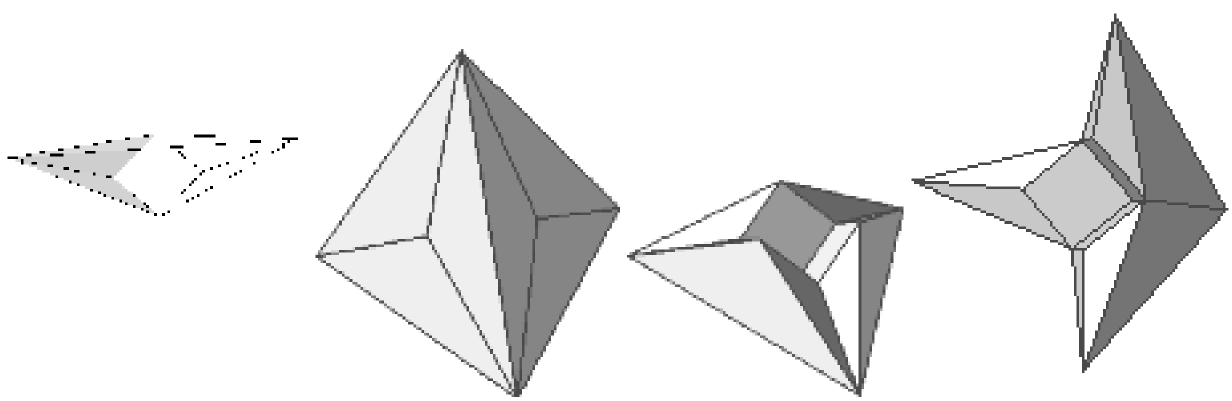


Figure 3. *Tetra Hold, tetrahedron/cube.*

Figure 4 presents a considerable step up in complexity. The exterior is a cuboctahedron displaying concave edges, while its hollow presents as a stella octangula, pictured on the right. The same exterior cuboctahedron appears in Figure 5 with a marked difference in its interior geometry that results in a space defining a great stella octangula.

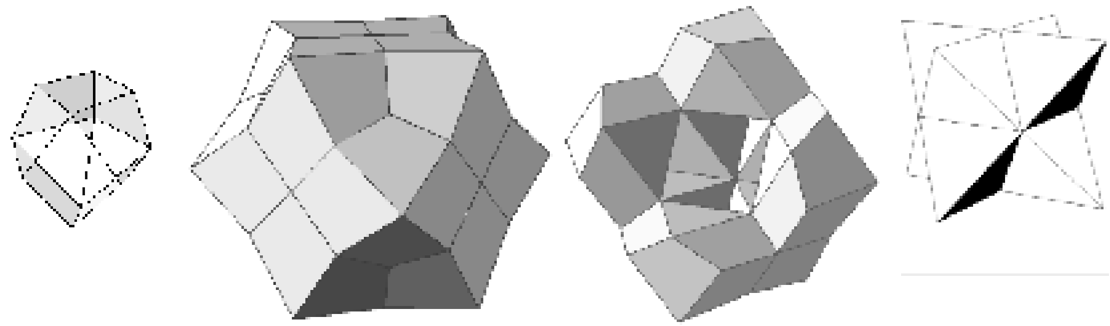


Figure 4. *Boxed Astroid, cuboctahedron/stella octangula.*

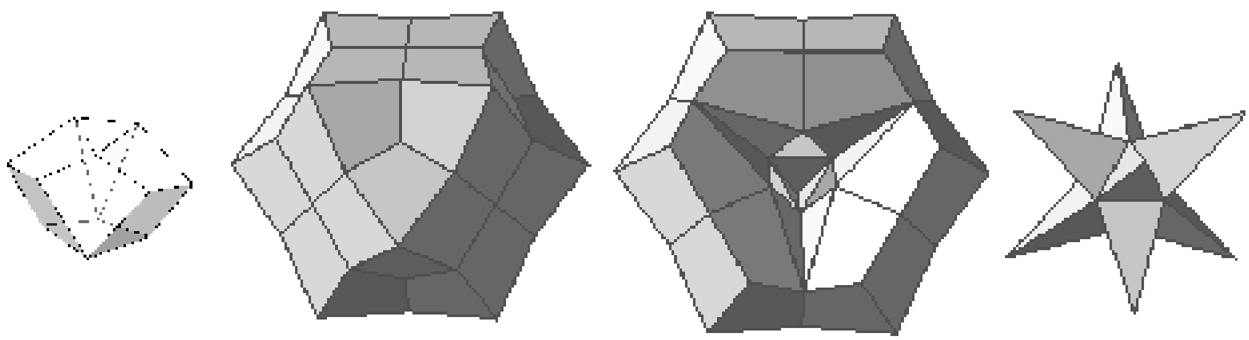


Figure 5. *Great Boxed Astroid, cuboctahedron/great stella octangula.*

Pentagonal Symmetry

Of the Platonic solids pentagonal symmetry manifests in the dodecagon and the icosahedron. Five-fold symmetry materializes in the pentagonal faces of the dodecahedron and in the five-way vertices of the icosihedron. Both figures harbor an intriguing relationship to the triacontrahedron (Figure 6 left). As a Catalan solid it is the dual of an Archimedean solid, the icosidodecahedron. Consequently its thirty rhombic faces (Figure 5 left) may be grouped to correspond to both the dodecagon and the icosahedron. The rhombs deploy into twelve five-way meetings of the narrow vertex of the rhombs or into twenty three-way fittings of the wide vertex of the rhombs. Dividing the rhombs lengthwise will demarcate the surface of the triacontrahedron into twenty equilateral triangles (Figure 6 middle) and so corresponds to the icosahedron, while dividing the faces widthwise creates twelve regular pentagons (Figure 6 right) and so corresponds to the dodecagon.

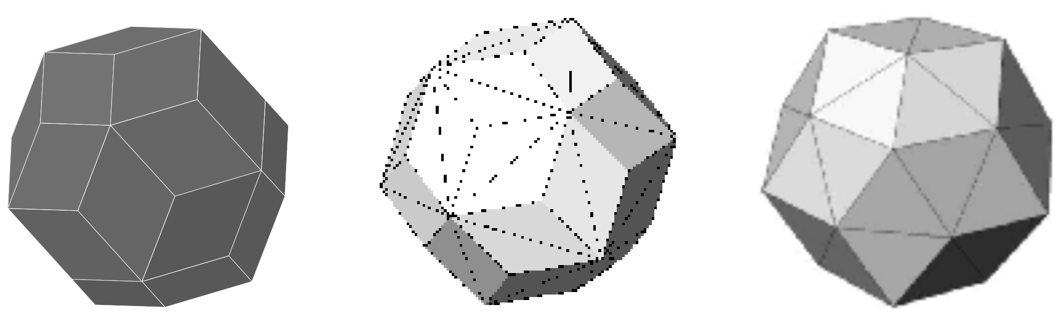


Figure 6. *(left) Triacontrahedron, (middle and right) Triacontrahedron with split faces.*

Figure 7 has an icosihedron embedded within a triacontrahedron. The sculpture comprises twenty modules each of which encompasses a three-way vertex of the outer solid. Figure 8 features a dodecahedron at the center of a triacontrahedron. This sculpture's twelve modules replicate each of the five-way vertices of that latter solid.

The pentagonal solids require more modules: twelve, twenty, thirty and potentially sixty units may be needed to build a single sculpture. Consequently, this symmetry group provides sources for increasingly intricate and detailed forms. In addition, these forms reference imagery, such as the allusions to flowers and stars in Figures 9 and 10.

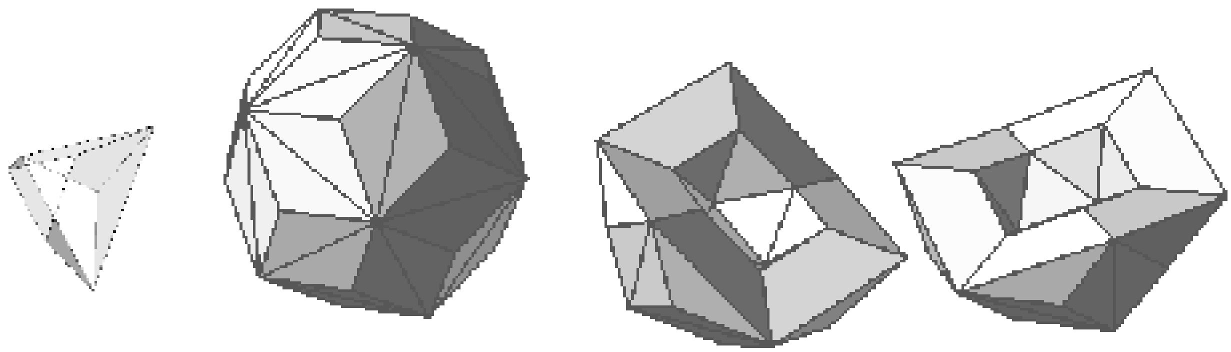


Figure 7. *Triacontrahedron/Icosahedron.*

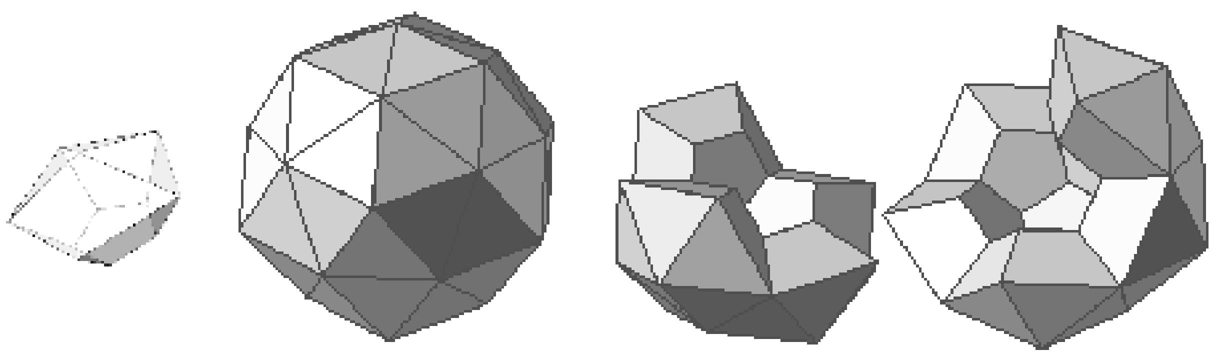


Figure 8. *Triacontrahedron/Dodecahedron.*

The almost pedestrian simplicity of the modular "blocks" can belie the potential intricacy of the end sculpture. The sculptures below exemplify the conceptual contrast between the part and the whole. The surprise result of the diamond-like modules in Figures 9 and 10 are two variations of the icosidodecahedron, the Archimedean truncation of the dodecahedron. Enhancing visual interest are the concave and convex faces in consort with their concave and convex edges. When split these sculptures keenly evince the effect of discovering the sharp crystal structures upon opening geodes (Figure 11). The spiky penetrations of the great stellated dodecahedron in Figure 9 and the stellated dodecahedron in Figure 10 combine with the incursions of the sculpture's inner vertices to generate this effect.

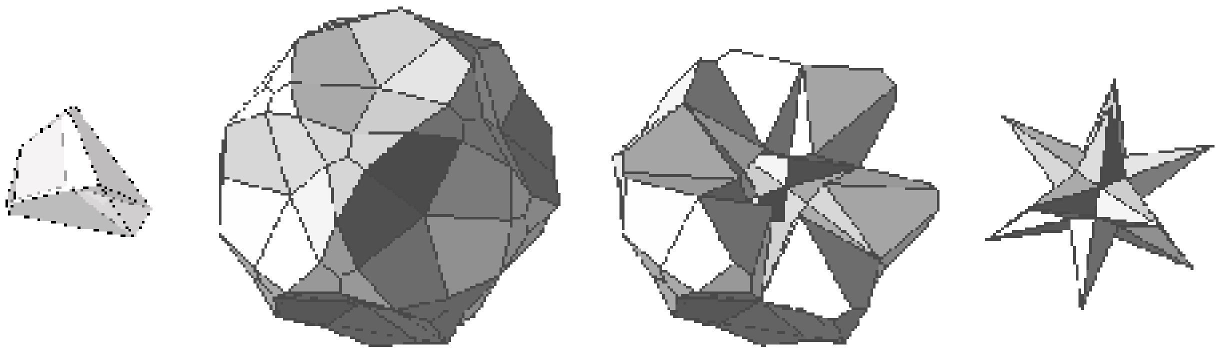


Figure 9. Flowered Geode, *modulated icosidodecahedron/great stellated dodecahedron.*

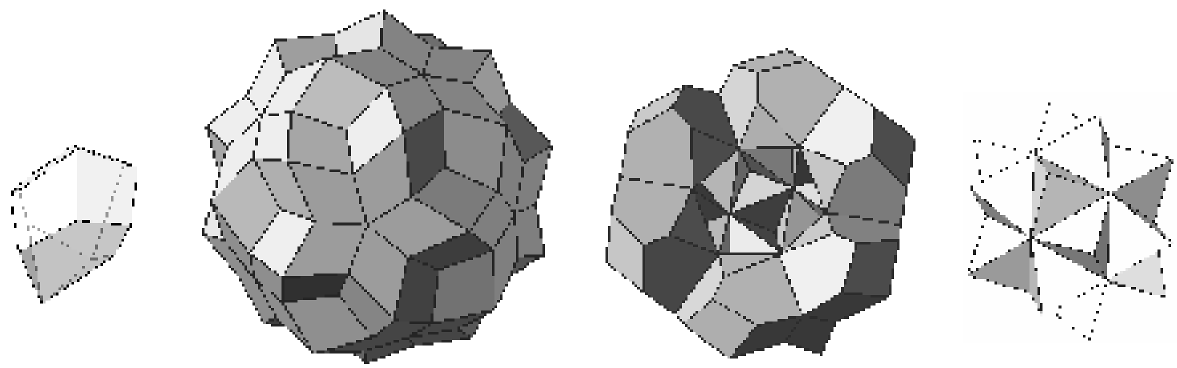


Figure 10. Starry Geode, *dodecadocahedron/small stellated icosihedron.*

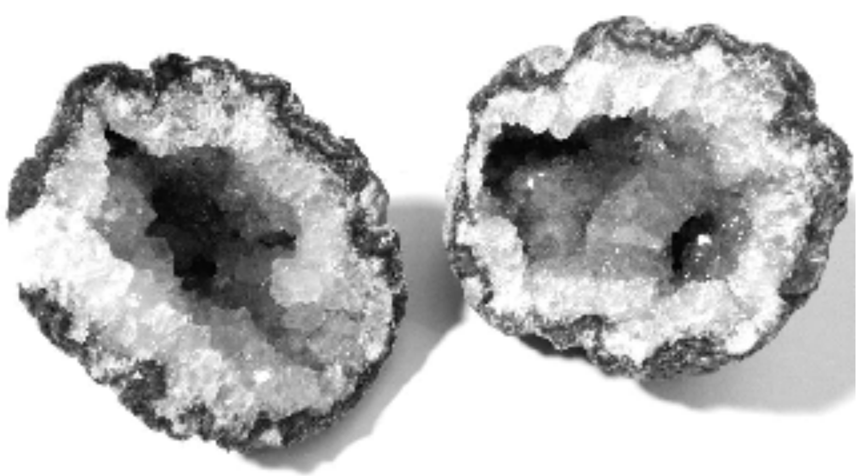


Figure 11. *Quartz geode.*

Sculptural Development

As the subject of art objects spherical polyhedrons are typically defined as structured surfaces, rather than as the more sculptural interaction of masses and volumes. The sculptures photographed below represent earlier investigations into this theme. Figure 12 depicts two sculptures, entitled Geods, with tank ends mounted on the hexagonal faces of a truncated tetrahedron and the octagonal faces of a truncated cube.

Figure 13 reveals the source of this imagery: the Neolithic stone balls found in Scotland and dating to the period of Stonehenge.

Breaking Bonds in Figure 14 sits in the central courtyard of the Chem-Life Complex at the University of Illinois. Its configuration into the truncated icosahedron references the carbon 60 molecule studied in the Chemical Engineering building. The chain of modules broken from the largest component bears strong resemblance to the structure of a cholesterol molecule studied in the organic chemistry building. The smallest component is a single module, serving both as a stool and an image of the graphite molecule as might come under scrutiny in the physical chemistry building.

Figure 12. *Geods, steel, 36" d. and 32" d. 1992.* **Figure 13.** *Neolithic carved stone ball, 2.7" d. (photo - http://www.bbc.co.uk/ahistoryoftheworld/objects/iZUovGvIRyyivoCUauZjxw)*

Figure 14. *Breaking Bonds, stainless steel and copper, spherical component 83" d. 1994.*

The current series of Plato's Blocks comprises a more extended and concentrated investigation of similar sculptural concerns.

The series has its source in a series of designs for landscape features for the Highlake Sculpture Garden in West Chicago, IL. Those designs called for polyhedrons of linear construction using landscaping timbers as structural units. Plantings on the interior of these polyhedrons would then fill the interior and twine on and between the beams to create a ball of foliage. Eventually, the landscaping idea left the table, but the polyhedral structures continued to present sculptural possibilities, in that the thickness of the beams allowed only small, although deep, penetrations through the polyhedrons' faces. As represented by the beams the polyhedral edges, effectively became masses and the polyhedrons became toroidal.

Figures 15 and 16 demonstrate two toroid versions of the dodecahedron, where the edges have turned into masses and the faces into negative frustums. As seen in comparison between these two figures, the greater the thickness of the beam in relation to the radius of the polyhedron, the smaller the holes between the beams.

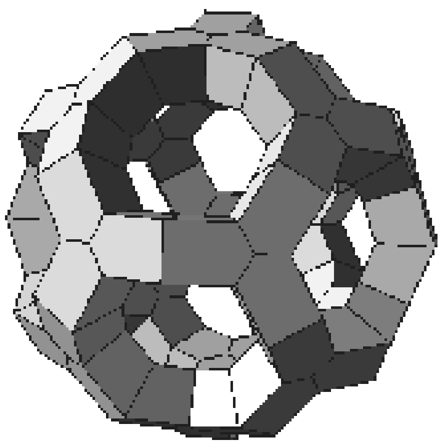


Figure 15. Fat Struts: Toroidal dodecahedron, v1.

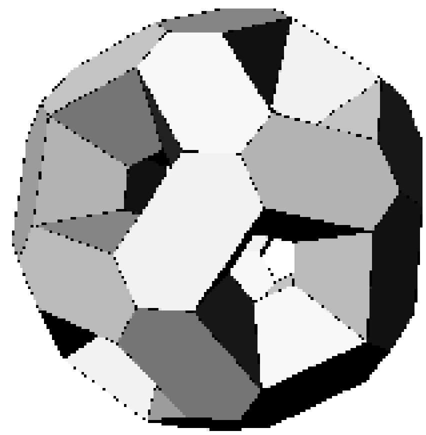


Figure 16. Fat Struts: Toroidal dodecahedron, v2.

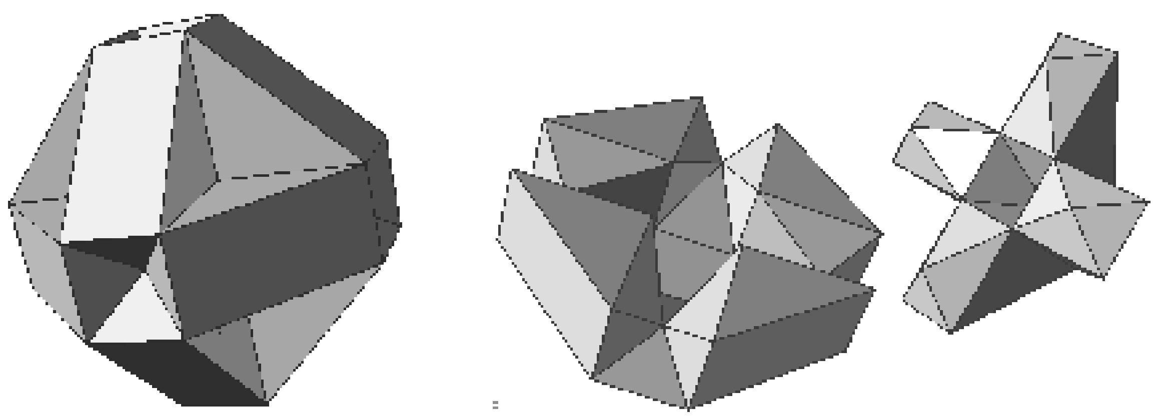


Figure 17. Fat Struts: Boxed Star v1

Figure 17 illustrates an octahedron at the stage where the beam has widened to the point where no holes remain and the structure is no longer a toroid. The form is losing much of its status as an octagon and is taking on the characteristics of a small rhombiccuboctahedron, albeit with half of its faces deepened into pyramids. Once the beams were mitered so that they fit together a pleasant surprise awaited the sculptor: a hollow core shaped as a stellated cube.

Shortening the timber so that the exposed face of the strut is a square would have completed the transition of the octahedron into its rhombic derivative. This is the case for the dodecahedron in Figure 18 below. With the outer face of the struts square the dodecahedron becomes a rhombicdodecahedron.

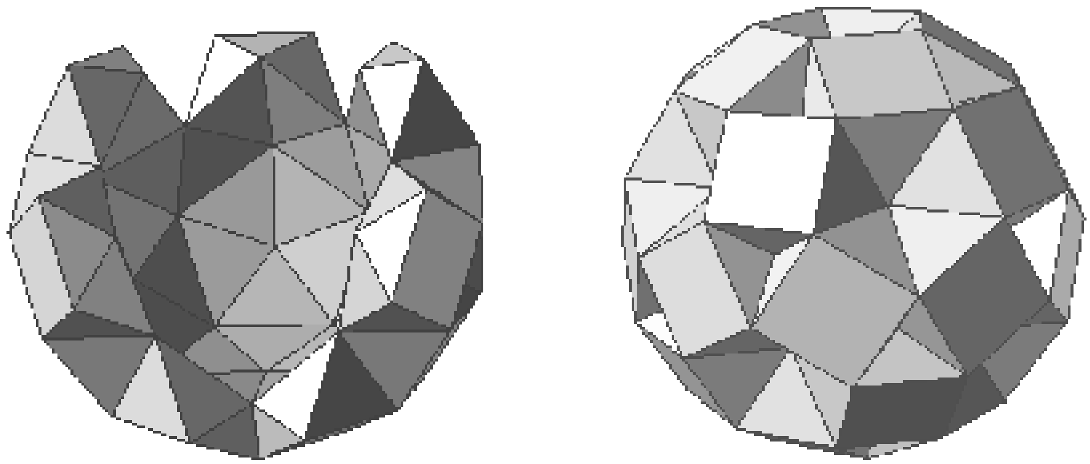


Figure 18. _Chambered Rhombicdodecahedron._

Flanking the struts into side-by-side pairs along the edges of a dodecahedron revealed the formation of a triacontrahedon inside the structure. The interior triangles seen on the right of Figure 19 are paired into the rhombic faces of the triacontrahedron. This demonstrates that the internal spaces are usually duals. In this case the triacontrahedron is a dual of the icosidodecahedron and that Archimedean solid is implicit in the pyramidal faces of this sculpture.

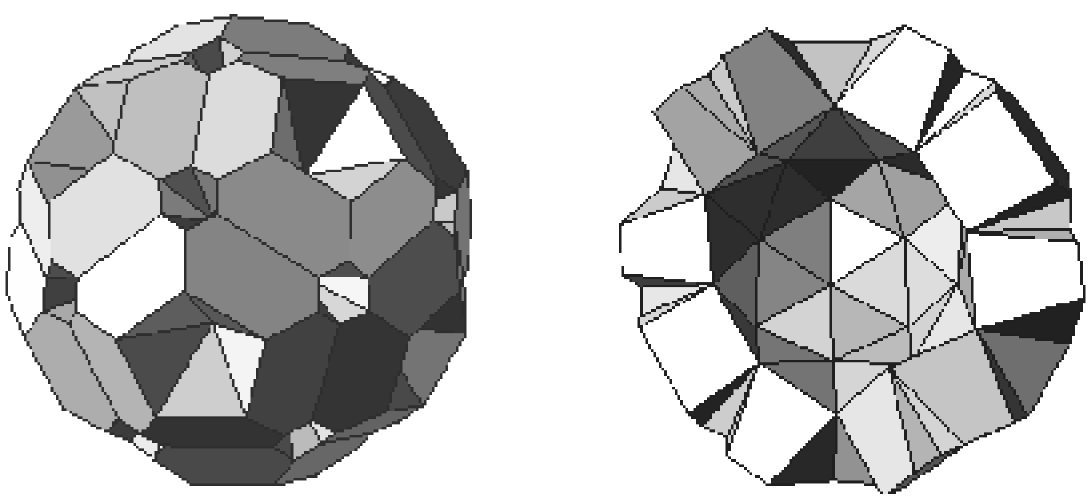
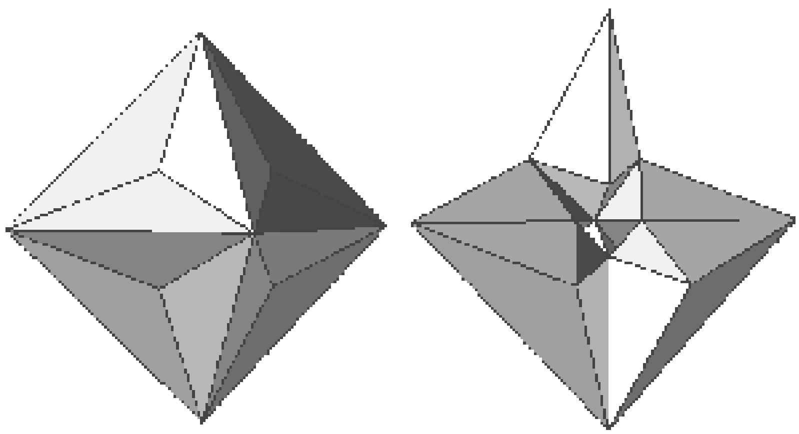


Figure 19. _Chambered Rhombicdodecahedron v2._

Similar surprises appeared when the structural beams were rotated 45° along the long axis so that one edge of the beam projected outward and the opposite edge projected inward. In this case the octagon remains clearly an octagon, but whose interior is a stella octangula, as depicted in Figure 20. With the inner edge of its struts truncated into faces, the interior transforms into a rhombic dodecahedron, a Catalan solid (Figure 21). Note that Figures 1, 2 and 3 also use this method of rotating the strut.



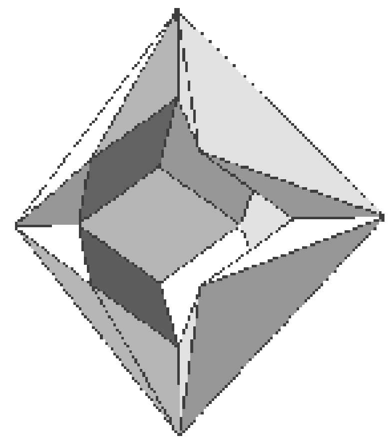


Figure 20. *Fat Struts: Boxed Star v2* **Figure 21.** *Chambered Octahedron*

Applying this same structural device of the beam edges turned outward to other Platonic solids created yet more stellated cores. Figure 22, for example, reveals a similarly constructed dodecagon to yield a stellation of the dodecahedron.

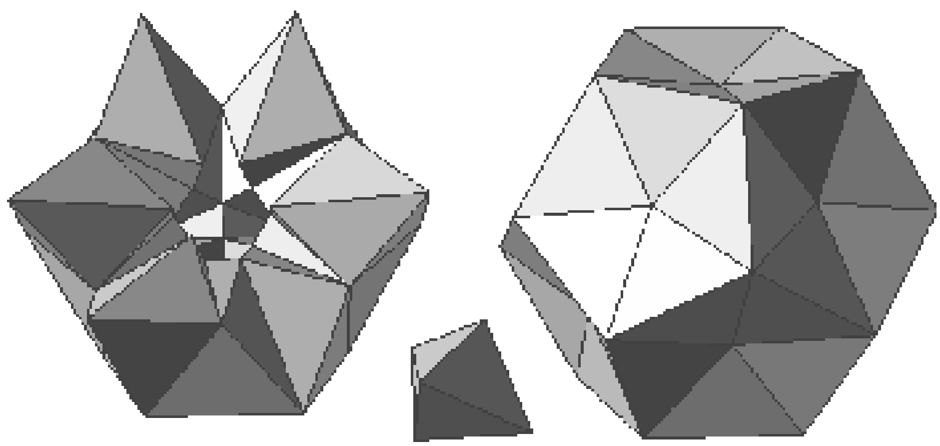


Figure 22. *Star Chamber.*

Conclusion

Building polyhedrons from landscaping timbers has offered this sculptor an oblique gateway into the possibilities for exploring the interaction of masses and volumes within the elegant symmetries of the spherical polyhedrons. Not just surface structure, but the deep penetration of spaces, gains purchase in art using these mathematical objects as their subject. Opening up the interior of spherical polyhedron also opens up the opportunity to use these figures to exposit a central tenet of sculptural: the negative volumes of the sculpture have form just as truly and just as powerfully as the filled masses.

Sketching in Four Dimensions

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Abstract

Combining 19th century explorations of dimensions beyond three with 21st century graphic drawing programs opens new avenues of design possibilities. This paper will attempt to provide an introduction to the 2-D design possibilities that exist when four-dimensional objects are drawn and manipulated with modern dynamic software packages. It begins with a brief introduction to the concept of multiple dimensions and then proceeds to specific examples and the methods used to obtain them. For the mathematical artist these concepts can lead to a whole new world of design possibilities.

Imagining the Fourth Dimension

The concept of four dimensions seems intuitively impossible to most people the first time they encounter such an idea. Yet scientists and mathematicians regularly talk in terms of multiple dimensions of four and beyond.

The initial reaction may range from “How is that possible?” to stronger opinions regarding the sanity of anyone who suggests the topic. This is not helped by the exaggerated claims of science-fiction writers who use extra dimensions to explore fantastic other-world scenarios beyond rational belief.

But four and more dimensions have intrigued mathematicians since the 19th century when they were studied by Möbius, Schläfli, and Riemann among others. The concept excited the imaginations of the general public, as well. One particular advocate was Claude Bragdon, an architect, ornamentalist, theatrical set-designer, philosopher, and prolific writer who authored two books detailing the geometry and its application to ornamentation in 1913-15.

Specifically, his book “Projective Ornament” laid out the principles of understanding four dimensions, methods to reproduce a four-dimensional object on a two-dimensional plane, and specific examples of how to develop from them whole systems of ornamentation. We can now combine these methods with dynamic graphic software programs to easily create whole new vocabularies of ornament. The purpose of this paper shall be to show specific examples of how this is done.

A Brief Introduction

Bragdon said in his book, “The concept of a _fourth dimension_ is so simple that almost anyone can understand it… if only he will not limit his thought of that which is _possible_….by his opinion of that which is _practicable_.” [1] With this in mind, lets briefly examine the basic structure and line of reasoning that leads to the fourth dimension.

A dimensionless point can be moved in one direction to form a line. The line can be moved along the plane to form a square, and a square can be raised away from the plane in the third dimension to create a cube. As illustrated in Figure 1, we do this without any confusion when illustrating a 3-D cube in a 2-D drawing.

Now, *imagine* that there is an additional dimension orthogonal to the three already mentioned, and move the cube a unit amount along that dimension in all directions. The result is called a hypercube, a four-dimensional object which cannot be physically realized in three dimensions, but *can* be illustrated by various means in lower dimensions. One of those means is by drawing just the wire-frame outline in 2-D. See Figure 1.

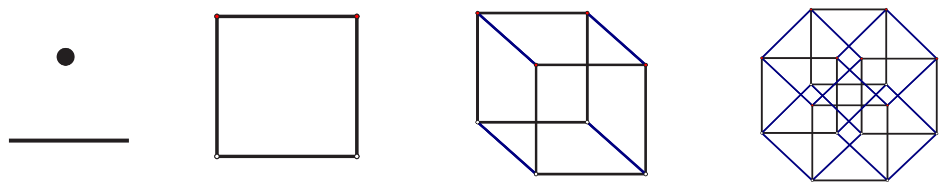

Figure 1: *Point, line, square, cube, hypercube*

The last picture (the hypercube) is a two-dimensional drawing of a three-dimensional representation of a four-dimensional object. Because this object is not familiar, the drawing can be harder to recognize until we have had some practice. Eventually we see the eight intersecting cubes that make up a hypercube and it becomes more understandable.

This illustration is a vertex-oriented view of the hypercube. It is the most common view and actually originated in Bragdon's 1913 book, "A Primer of Higher Space". The hypercube can be drawn from other angles as well. Figure 2 shows other viewpoints of the same object.

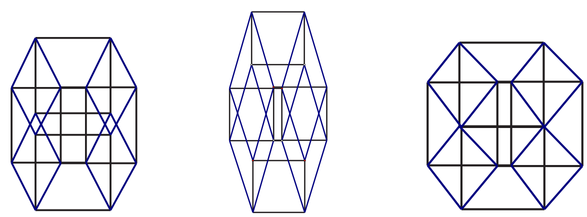

Figure 2: *A hypercube as viewed from different angles*

Of course any other three-dimensional object can also be extended into another dimension as well and then that new 4-D object can be represented in two dimensions. Figure. 3 shows a hexadekahedroid, or 16-hedroid. It is formed from sixteen tetrahedra that have been positioned uniformly in four dimensions. One tetrahedron is shaded for reference.

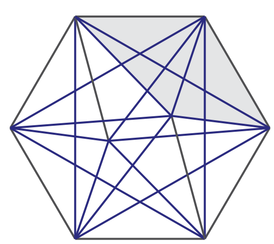

Figure 3: *A four-dimensional hexadekahedroid composed of 16 tetrahedra*

Sketching in Four Dimensions

The graphic drawing program entitled Geometers Sketchpad provides a valuable tool for manipulating objects because it maintains the integrity of their structure as they are stretched, skewed, or otherwise modified. Figure 4 shows a view of a hypercube before and after a simple movement of the red line to the right which elongates the structure.

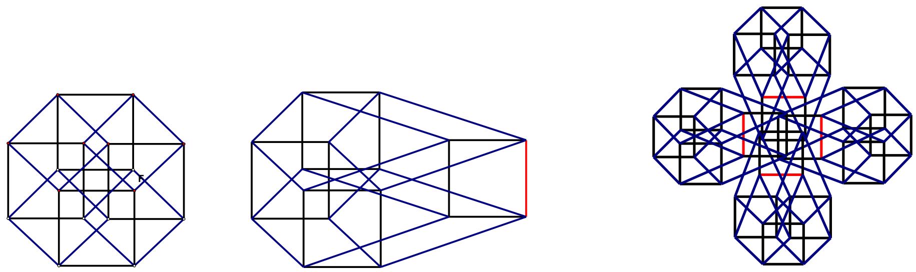

Figure 4: *A stretched and rotated hypercube*

When repeated, rotated and slightly overlapped, the new design in Figure 4 is created.

In theory, this is the type of manipulation that Bragdon used to formulate his designs. The next section will present a detailed example to illustrate the possibilities afforded the modern designer with dynamic software at her command.

Designing in 4-D

In Geometers Sketchpad the basic hypercube is stretched to give Figure 5. There are a variety of transformations available to use, but the first one will be dilation, to reduce the size so it can be repeated in a manageable fashion. Setting a point of dilation as shown to the right, the original figure is selected and a ratio of .7 to 2 gives the smaller figure (the illustration is not exactly to scale because of publishing concerns).

Next will come the rotation transformation. Setting a central point of rotation and then successively rotating by ninety degrees three times about that point yields the figure in Figure 6. Again, the scale has been adjusted.

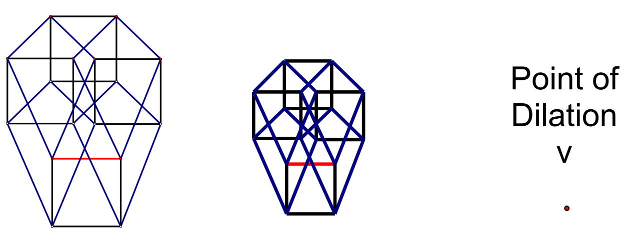

Figure 5: *Dilation of hypercube*

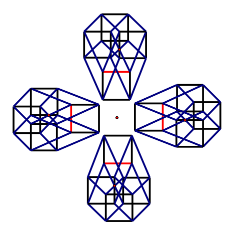

Figure 6: *Rotated hypercube*

Now the power of the software really comes into play. By simply adjusting the central point of rotation the spacing can be completely controlled. Moving the point *vertically only* causes the arms to overlap by varying degrees, even to the point of inverting the figures as shown in Figure 7.

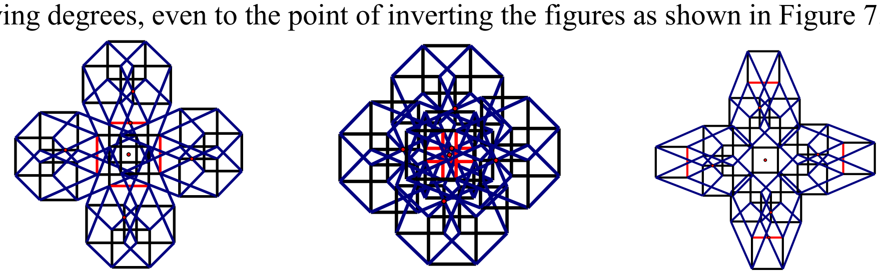

Figure 7: *Overlapping figures formed by vertically adjusting the center point*

Moving the point of rotation *horizontally* introduces a skew motion Figure 8 left, so that moving it both horizontally and vertically results in Figure 8 right. Again the arms have been inverted.

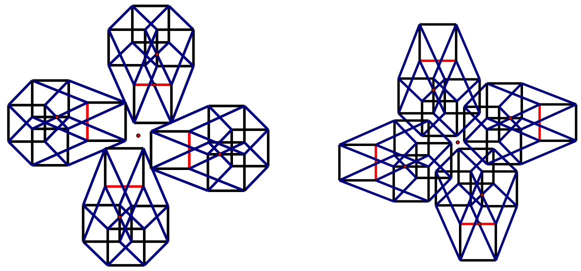

Figure 8: *Horizontal and diagonal adjustment*

Similar manipulation easily leads to more designs, as shown in Figure 9.

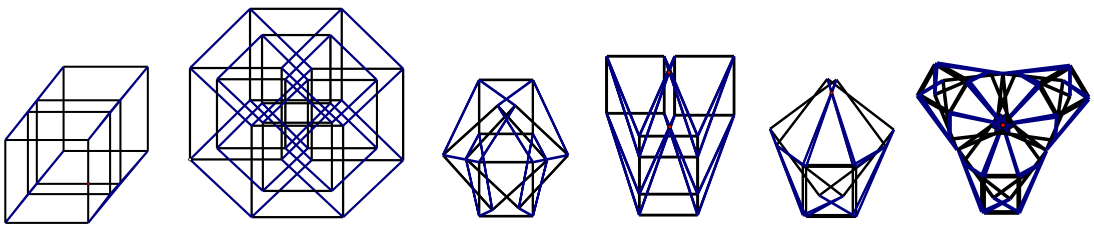

Figure 9: *Hypercube manipulation in Geometers' Sketchpad*

From Design to Artwork

Design enhancement is the other half of the artists' toolkit. Suitable drawings should be imported into coloring and rendering software. One of the simplest, yet still effective, is the Paint program available on almost every Windows installation. Color fill, cropping and background deletion are just some of the commands available in Paint. A more advanced package like Photoshop Elements offers additional features such as layering, filters, color duplication and selection, and numerous other possibilities. Figure 10 shows the input sketch and the final output from Photoshop that was used for the author's entry to the 2012 Bridges Conference math/art exhibit.

It is hoped that this paper can offer some instruction and inspiration for other explorations.

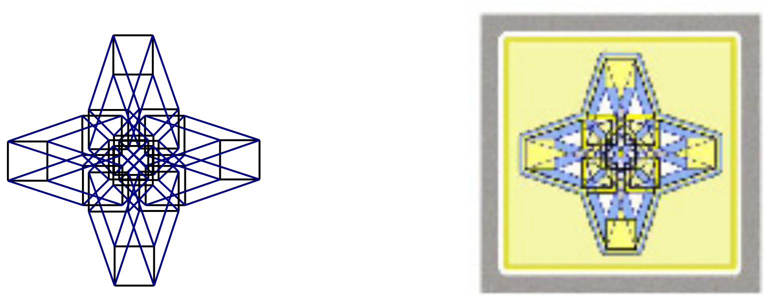

Figure 10: *Enhancement of design sketch*

References

[1] Claude Bragdon, *Projective Ornament*, The Manas Press, 1915 Reprint by Kessinger Publishing

[2] *Geometers Sketchpad* software, available at http://www.keypress.com/

[3] Susan McBurney, *The Projective Ornament of Claude Bragdon*, Joint Mathematics Meeting, 2012

Curves, Curved Surfaces, Hyperbolic Surfaces

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Sea shells always fascinated me with their various shapes and delicate colors. I drew them, made linoleum prints of them and painted pictures of them.

Figure 1: blue nautilus, linoleum print, colored pencil. This is an abstract representation of a nautilus interior.

Figure 2: large red nautilus, lino print and colored pencil

Later, in high school, I found that all the beautiful curves involved in nature were mathematical in some way and thus very predictable; that math was all about understanding structures and saying something about the regularities involved.

The nautilus shell follows a logarithmic spiral, a curve that spirals outward from a center in arcs progressively further away from the previous layer. In polar coordinates (r, θ) the logarithmic curve can be written as

$$r = ae^{b\theta}$$

where a, b are positive real numbers.

For the most part, these shells are positively curved, i.e. if you cut out a small piece of shell, it will look like a small circular neighborhood of the North Pole. It's not flat, rather it's like a small round hill, i.e. it's positively curved. Locally, it's spherical.

A hyperbolic surface is characterized by the property, that locally (in the vicinity of each point) it looks like a saddle. So it is not a flat surface. You will only get a very crude approximation, if you try to model it out of flat pieces of cardboard. This dilemma was already noticed centuries earlier, when the first map makers tried to faithfully depict the globe (mathematical sphere) on a flat sheet of paper. Trying to do that introduces distortions that will either make the land masses near the North and South Poles appear larger or the regions near the equator appear smaller than they are relative to land masses elsewhere. In essence we have discovered spherical, flat and hyperbolic geometry.

Figure 3: large blue shell, gouache and acrylic.

Figure 4: red snail, gouache and acrylic

Figure 5: blue snail, gouache and acrylic. Figure 6: two snails, gouache and acrylic. Figure 7: broken whelk, gouache and acrylic. The figure shows when the snail shell is broken, you can see its structure.

Another way to distinguish flat, spherical and hyperbolic surfaces is by the different forms the parallel axiom takes:

You start with a straight line g and a point P not on that line.

In the Euclidean plane, there is exactly one line through P not intersecting (parallel to) g.

On the sphere any straight line (in this case a great circle) through P will intersect g.

On a hyperbolic surface there are infinitely many straight lines through P and not intersecting g.

The straight lines on a hyperbolic disk are either straight lines through the center or are circles which are perpendicular to the boundary of the disk. So it's easy to see (in the next picture) why there are infinitely many parallels through a point to a given straight line not passing through this point

Previous Work

There are many mathematicians and artists, who have depicted and constructed these surfaces. I owe a debt of gratitude to all of them. Maurits Cornelis Escher (1898-1972) was a graphic artist who among other aspects of his work, was fascinated by the hyperbolic disk. He did wonderful woodcuts, the circle limit series for example, depicting tessellations of the hyperbolic Poincare disk. Richard Serra (born November 2, 1939) is an

American minimalist sculptor working with large-scale assemblies of sheet metal. Many of his large metal sculptures are bands of hyperbolic surfaces. Despite their huge size and weight, their shape makes them look light and elegant. Carlos Sequin [2] is a professor of Computer Science at UC Berkeley. He uses three dimensional printing techniques for his smaller models. He also makes large metal sculptures of minimal surfaces. Eva Hild [3] models large hyperbolic surface sculptures from clay.

The idea of crocheting hyperbolic surfaces was first put in practice by mathematicians/artists: Hinke M. Osinga & Bernd Krauskopf [1], and Daina Taimina [4] at Cornell University. I learned about it through her husband David Henderson, who was my Ph.D. advisor. This idea was also taken up by Christine and Margaret Wertheim [5], from the Institute For Figuring, who saw a connection to sea creatures and in collaboration with many enthusiastic women crocheted coral reefs, which were widely exhibited in the US and overseas. Hyperbolic surfaces occur frequently in nature: in the shapes of corals, salad leaves, leaves in general, blossoms, jelly fish, etc.

There are also buildings based on hyperbolic surfaces, An early example was the tower of Shukov [6] in Nizhni Novgorod. If you take a curved, loop of wire and dip it into soapy water, you will get a soap film modeling a "minimal surface", i.e. a surface of minimal area with prescribed boundary. It appears that such surfaces inherently more stable from an engineering standpoint.

Crocheted hyperbolic surfaces

So what is a relatively "natural" way of creating these surfaces? If you crochet a small flat disk by working in a spiral from the inside out, and then, rather than making the right number of stitches around the perimeter, you make a few more than you actually need, this will cause the surface to become wavy around the perimeter. Eventually the waves become significant and you have created a saddle. If you keep increasing the number of stitches, say for every 8^{th} stitch you make an additional stitch (into the same place where you made the eighth), then this will create a hyperbolic surface. If you make fewer than the necessary number of stitches, in other words you leave a gap, the surface will be spherical, if you make just the right number of stitches, then the surface will become flat.

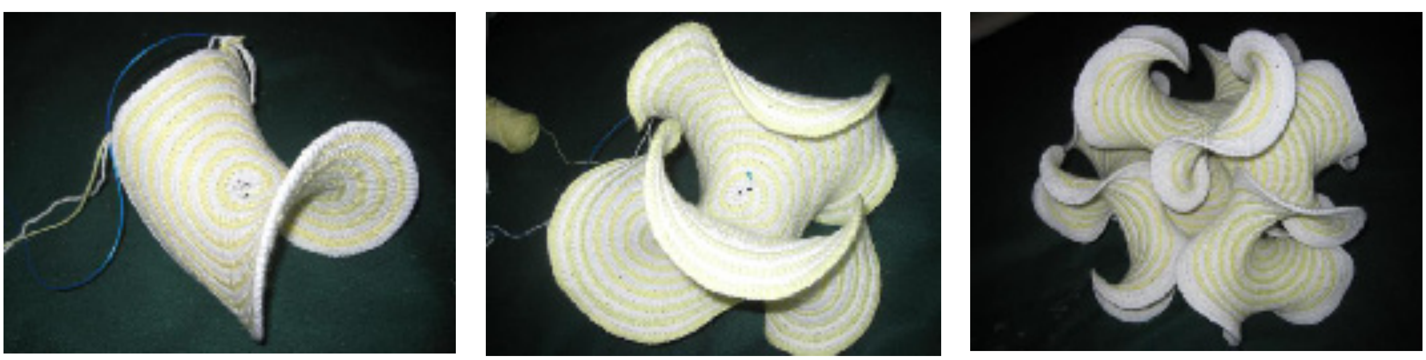

Figure 8. You can see the initial flat stage of the hyperbolic surface, in a small disk around the starting point. At this later stage, the saddle shape is obvious. The shape eventually turns to a hyperbolic disk, double blossom stage. Finally, it becomes a hyperbolic disk, final stage.

The evolution of a hyperbolic surface:

It starts out round and flat. Then it curves slightly around the perimeter. It takes on the shape of a potato chip (my mother-in-law's insight!). Then it becomes a saddle.

After further rows of crocheting, it will progress to this almost double blossom stage. Later it will take on its final shape. I have not tried to go much further. In my experience, the shape will be more and more dense and making it harder and harder to see through to the middle.

Figure 9: red-green hyperbolic disk and Figure 10: pink and red blossom

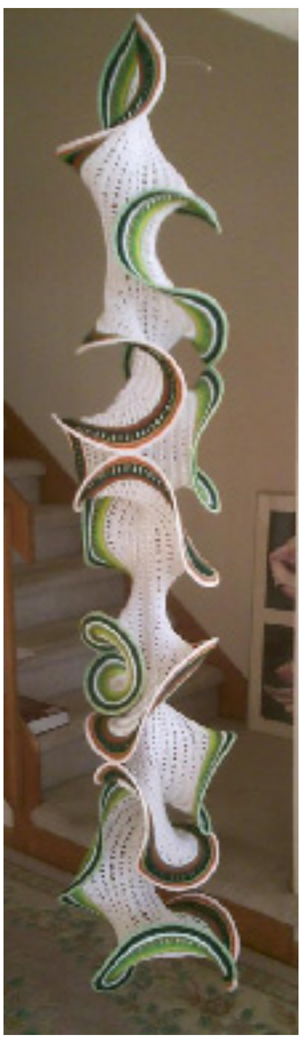

Figure 11: asymmetric green algae.

My aim was to make hyperbolic crocheted surfaces that kept their shape and would not flop, as crocheted objects usually do. After some attempts with willow branches (they broke) and clothes line (too thick, but generally the right idea), I crocheted in weed whacker line, which works perfectly. The surface becomes firm, but not hard, won't weigh much, doesn't hurt when you bump into it and looks pretty! The technique makes this something new, in between crocheting and basket making.

So far I have made three basic types of surfaces, all with pretty much constant curvature: the hyperbolic disk, the blossom, which has a long stem, and the algae, which consists of two hyperbolic half planes attached to each other, a simplified version of that is the screw.

You can also crochet with holes, then the outcome can be used as a lamp shade using an energy saving bulb (caution fire hazard). But since light and shadow are displayed very interestingly on hyperbolic surfaces, it is very worth while to find a way to use the surfaces in conjunction with light sources.

Other, more random shapes, such as combinations, are no doubt possible. Making them asymmetric and just simply irregular will open a whole new vista of sculptures. One might also make objects that are more strongly curved in some places than in others.

Questions for the future:

Why are lettuce leaves hyperbolic? If you spread a leaf out, it's curved fan shape becomes obvious. Is it, because its biology prefers it to have as much surface area as possible for a given amount of space? Maybe in order to take advantage of more exposure to sunlight and oxygen? The same goes for algae.
Is this a principle, that might be useful for human habitation aswell? For bridges, roofs, plant cultivation etc.?

Figure 12: white hyperbolic disk with holes.

References:

[1] Hinke M. Osinga & Bernd Krauskopf, Crocheting the Lorenz Manifold, *The Mathematical Intelligencer* 26(4) (2004) 25-37

[2] Sculpture Designs by Carlo H. Séquin, Scherk Collins Toroids, http://www.cs.berkeley.edu/~sequin/SCULPTS/SEQUIN/scherkcollins.html http://www.cs.berkeley.edu/~sequin/ART/Science_Sculpture/

[3] Eva Hild: Topological Sculpture from Life Experience, http://math.albany.edu/math/pers/artmath/art/eh.html

[4] Daina Taimina's webpage: http://www.math.cornell.edu/~dtaimina/

[5] Webpage of the Institute For Figuring: http://www.theiff.org/

[6] Shukov Tower in Novgorod: http://en.wikipedia.org/wiki/Hyperboloid_structure

[7] Muenchen Olympic Stadium: http://en.wikipedia.org/wiki/File:Olympiastadion_Muenchen.jpg

Figure 13: various hyperbolic surfaces at the Museum of Wisconsin Art.

Modular knots from simply decorated uniform tessellations

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Abstract

Knots and links are a common theme in two-dimensional artworks that span cultures and time periods. Knots are found as Greek, Roman, and Coptic decorations, reaching a high point with the Celts in the 7th century AD. A method of creating complex knots and links is presented which uses a modular approach that begins with a k-uniform tessellation. Each regular polygon is decorated with a simple motif that that has arcs connecting uniformly spaced points on the sides of the polygons. A variety of complex knots and links can be created using this procedure. Examples of visually interesting knots and links created using this procedure are presented.

1 Introduction

Knots have also been used for nearly two thousand years as decorative elements and are common in many cultures [1]. Spiral, key, and interlace patterns have been used for thousands of years as decorative elements in mosaics and other artworks. Celtic artists were masters at using plaits to decorate otherwise simple objects. While the process the Celts used for generating knots has been lost, many believe knots were created using an underlying square lattice decorated with lines connecting diagonally opposite corners.

Uniform tessellations, where the number and order of polygons meeting at a vertex remains constant throughout the tessellation, are a common decorative element for planar surfaces. The simplest uniform tessellations are the tessellations by squares, regular hexagons, and regular triangles. There are eight other uniform (more precisely 1-uniform) tessellations that form the eleven Archimedean tessellations.

The author has previously described a technique for creating interlace patterns by decorating the polygons in such tessellation using a simple motif comprised of Bézier curves [2]. Each n-gon is decorated by subdividing its sides and placing d uniformly spaced endpoints along each side, resulting in nd endpoints. Each endpoint can be assigned a unique integer $0, 1, \ldots nd-1$ starting with the first side clockwise from a vertex. A arc pattern in a single polygon can be described using the following notation:

$$(\alpha, \beta | \gamma, \delta | \varepsilon, \zeta | \cdots)$$

where $\alpha, \beta, \ldots$ represent the endpoint numbers of arcs in a the polygon. A total of $nd/2$ arcs made from simple cubic Bézier curves connect pairs of endpoints such that the tangent of each at the endpoints is

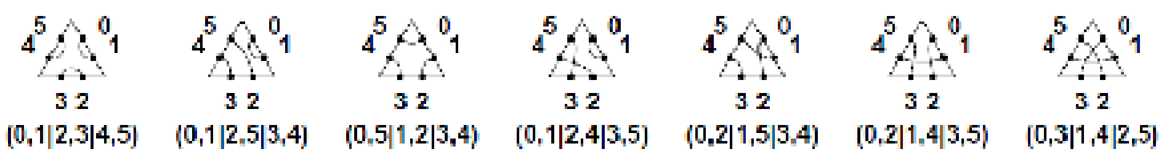


Figure 1: *Example tiles decorated with Bézier arcs. The seven geometrically unique crossing pattern motifs for decorating triangles using three Bézier arcs. Each triangle side contains two arc endpoints. Notation for each pattern is shown below the corresponding triangle. Such crossing pattern motifs can be constructed as long as the number of arc endpoints is even.*

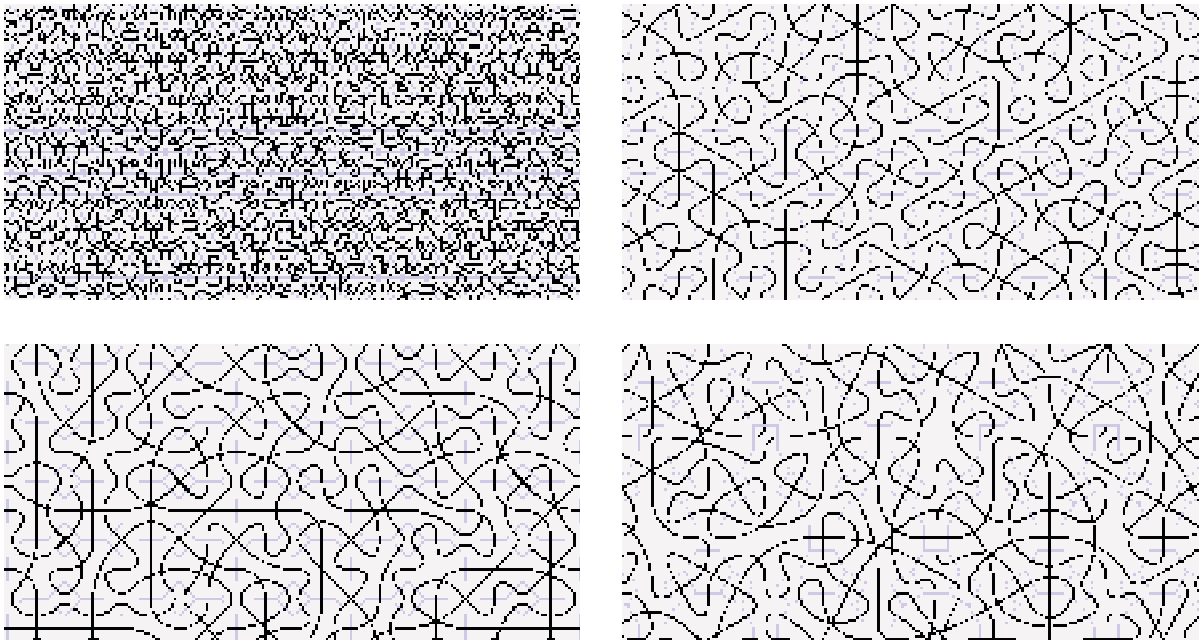


Figure 2 : *Examples of Bézier curve motifs decorating polygons comprising uniform tessellations. The upper-left figure is a tessellation by equilateral triangles (3,3,3,3,3,3) using two arc endpoints per side. The upper-right figure is a tessellation by regular hexagons (6,6,6) using one arc end point per side. The lower-left figure is a tessellation by regular octagons and squares (8,8,4) using one arc end point per side. The lower-right figure is a tessellation by regular dodecagons, hexagons, and squares (12,6,4) using one arc end point per side. The underlying tessellation is shown faintly in the background of each figure.*

perpendicular to the polygon edge. Examples of polygons decorated in this manner are shown in Figure 1. The number of unique motifs grows exponentially with the number of endpoints per side and the number of polygon sides [3]. In this manner, the curvature of the arcs at the boundary is continuous, giving a visually pleasing meandering pattern. An example using randomly placed motifs, shown in Figure 2, is suggestive of threads that wander through space in a semi-regular manner.

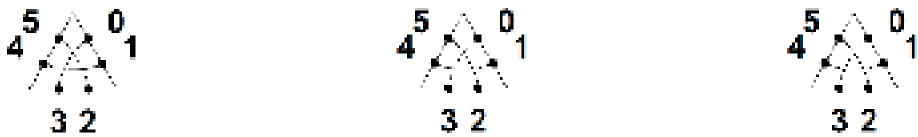
2 Methods

Knots can be generated in a modular fashion by treating the interlace patterns obtained from decorating polygons in a uniform tessellation with Bézier curves as threads that cross one another rather than simply intersecting. To indicate a crossing, the lower thread is broken into two segments near the crossing point, producing pattern commonly used in presenting knots using a knot diagram.

Finding the location where two arcs intersect requires finding the intersection point of two Bézier curves. A cubic Bézier curve is a parametric curve formed by the weighted combination of four control points, P_0, P_1, P_2, and P_3:

$$B(t) = P_0 t^3 + P_1 t^2 (1-t) + P_1 t (1-t)^2 + P_3 (1-t)^3,$$

where $0 \leq t \leq 1$. In this work, P_0 and P_3 are endpoints that lie on the edge of a polygon. The control points P_1 and P_2 are selected such that the curve is perpendicular to the edge of the polygon at points P_0 and P_3. The control points P_1 and P_2 can be selected such that the maximal curvature is also minimized along the curve. The intersection of two Bézier curves is nontrivial because in the general case, the curves may have up to six intersection points. However, since the arcs used are known to have a simple rounded shape, at most a single



$(0,3|1,4|2,5)[0/1,1/2,2/0]$ $(0,3|1,4|2,5)[0/2,1/0,2/1]$ $(0,3|1,4|2,5)[0/1,2/0,2/1]$

Figure 3: *Examples of knot sub-patterns. These over/under knot sub-patterns are based on the* $(0,3|1,4|2,5)$ *arc pattern shown in Figure 1. The arcs are labeled 0, 1, and, 2 based on the position in the parenthesized list; here arc 0 connects endpoints 0 and 3, arc 1 connects endpoints 1 and 4, and arc 2 connects endpoints 2 and 5. Note an intersection can result in either over/under crossing type. Thus a motif with k crossings can have up to* 2^k *crossing combinations (five others for this crossing pattern are not shown).*

point of intersection occurs between any given arcs. This intersection point is determined using a numerical algorithm.

The notation for an arc pattern given in the previous section can be augmented to describe a knot sub-pattern on a polygon as follows:

$$(\alpha,\beta|\gamma,\delta|\varepsilon,\zeta|\cdots)[A/B,C/D\cdots]$$

where $\alpha,\beta,\ldots$ represent endpoint numbers and A/B indicates arc A (from endpoints α to β) overcrosses arc B (from endpoints γ to δ) . Examples of such knot sub-patterns is shown in Figure 3. Note an intersection can result in either over/under crossing type, so that a motif with k crossings can have up to 2^k crossing combinations.

The number of crossing combinations results in nontrivial drawing of knot sub-patterns. This can be seen in the arcs shown in Figure 3. In some cases, an arc can be drawn as a continuous segment between endpoints, such as arc 2 (between endpoints 2 and 5) in the pattern $(0,3|1,4|2,5)[0/1,2/0,2/1]$. However in this example, arc 1 (between endpoints 1 and 4) must be drawn as three separate sub-arcs.

Bain described a process for creating Celtic knots by connecting diagonally opposite corners in square lattice decorated [4]. The dual of a lattice of squares results in new lattice of squares rotated 45° from the original, so that a diagonal line passing through vertices in the original lattice will pass through the midpoints of edges in its dual as seen in Figure 4. Thus, the technique described here can be though of as a generalization of such Celtic knots.

3 Results

Examples using this procedure were constructed on several tessellations. Figure 5 shows an example containing several knot patterns based on the tessellation by triangles (3,3,3,3,3,3) where each triangle is decorated with three arcs. Figure 6 shows an example containing a repeating knot pattern based on a tessellation by squares (4,4,4,4) where each square is decorated with four arcs. Figures 7 and 8 show examples containing repeating knot patterns based on tessellations by hexagons (6,6,6) where each hexagon is decorated with three arcs. Figure 9 shows an example containing a several knot pattern based on a tessellation by octagons and squares (8,8,4) where each polygon is decorated with arcs connecting edge midpoints. Figure 10 shows an example containing a several knot pattern based on a 2-uniform tessellation by dodecagons, squares, and triangles, (12,12,3;12,3,4,3) where each polygon is decorated with arcs connecting edge tri-points.

4 Discussion

Even with this very simple design element, a wealth of interesting knots patterns is possible. One reason for the visual appeal of the patterns is the arcs of adjacent tiles are not only continuous, but also have a continuous first derivative resulting in a visually smooth transition regardless of tile orientation. Minimizing the maximal curvature of the arcs within a triangle results in curves with graceful sweeps.

Future work involves improving the three-dimensional display of these knots. Another goal is to identify and characterize the knots and links present in a given pattern. Given a tessellation and motif family, it is unknown how many knot diagrams for a given knot are constructable for a given knot, this is of particular interest for knots that can be presented in a symmetric manner. It is also of interest to analyze knots by calculating invariants.

A method was described for constructing visually interesting knots in a modular manner based on k-uniform tessellations of the plane. These patterns can be useful in very large field architectural tilings.

5 Acknowledgments

Support for this work was provided by the Great Lakes Colleges Association as part of its New Directions Initiative, made possible by a grant from the Andrew W. Mellon Foundation. This work was also supported by a grant from the Hewlett-Mellon Fund for Faculty Development at Albion College, Albion, MI.

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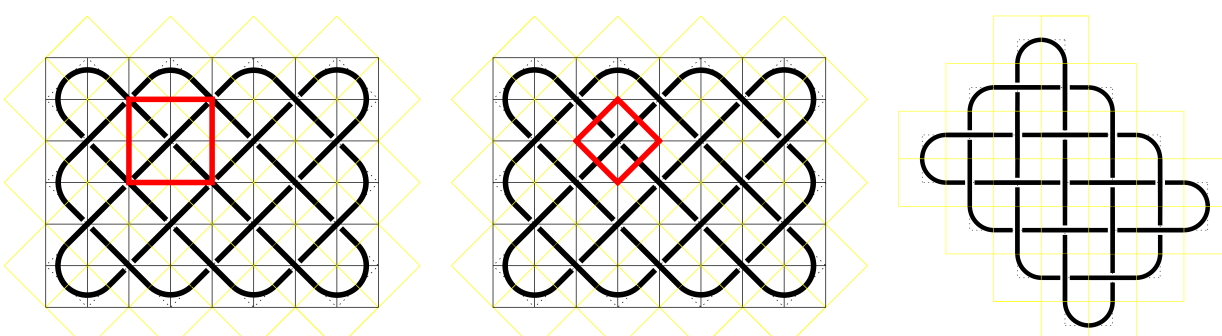

Figure 4 : *An example Celtic-style knot. Celtic knots such as seen in this figure can be considered a special case of the knot patterns described in this text. By considering the dual tessellation of the original square lattice and rotating by* 45°*, the rightmost pattern can be seen to be comprised of squares decorated with arcs connecting midpoints of edges.*

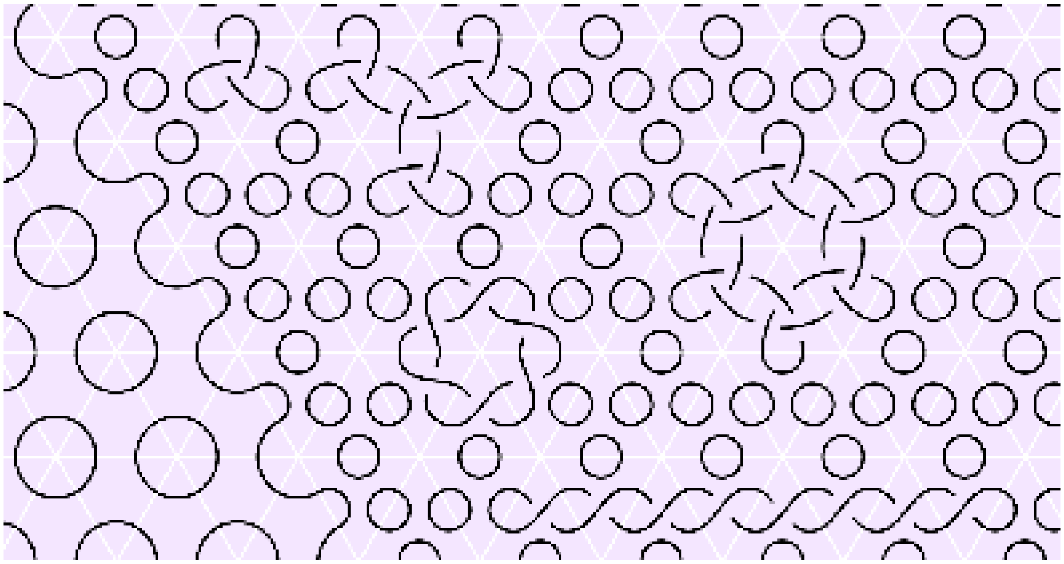

Figure 5 : *An example based on the tessellation by triangles. This figure shows an example containing several knot patterns based on a triangular tessellation (3,3,3,3,3,3) where each triangle is decorated with three arcs. Note that trefoil knots can be constructed in this manner.*

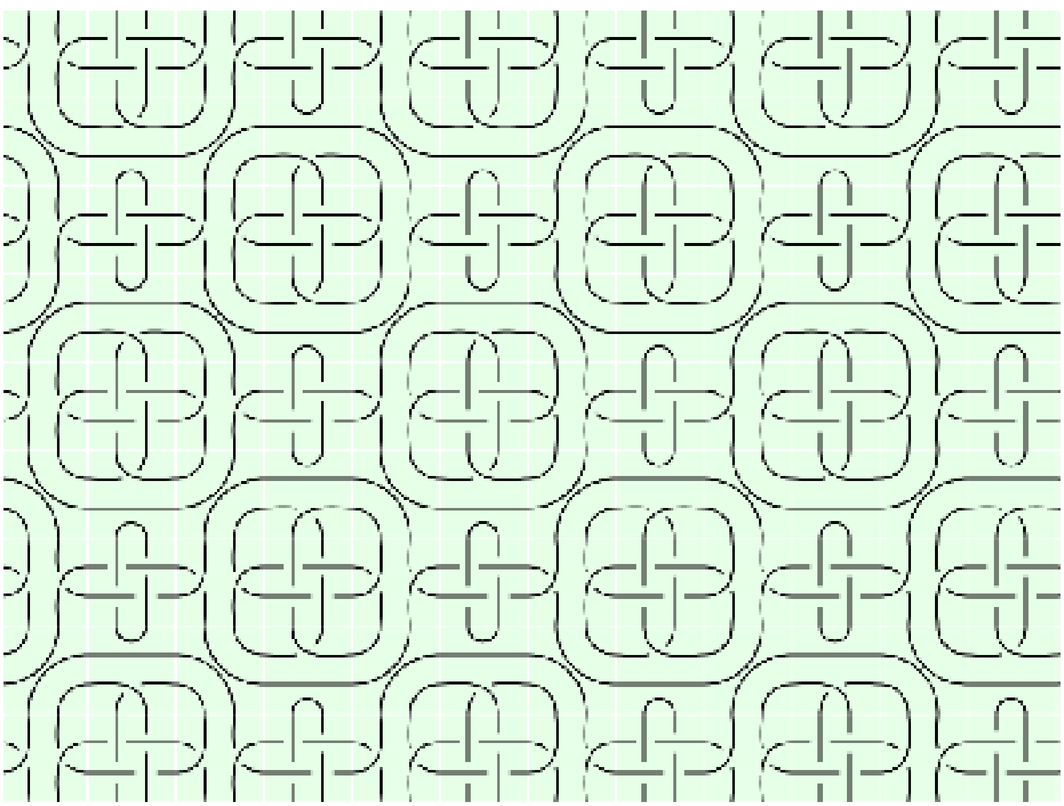

Figure 6 : *An example based on the tessellation by squares. This figure shows an example containing a repeating knot pattern based on a tessellation by squares (4,4,4,4) where each square is decorated with four arcs.*

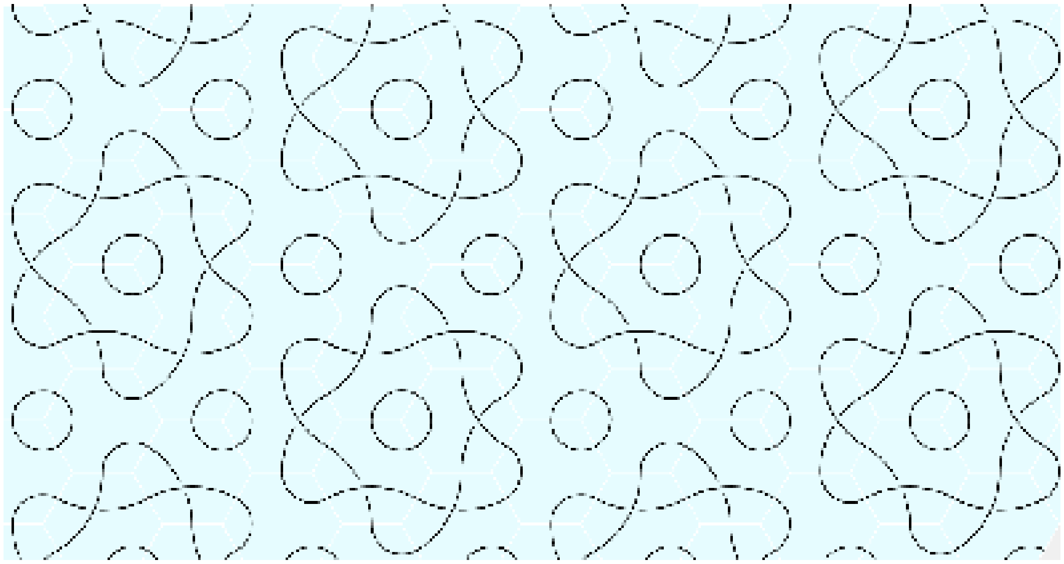

Figure 7 : *An example based on the tessellation by hexagons. This figure shows an example containing a repeating knot pattern based on a tessellation by hexagons (6,6,6) where each hexagon is decorated with three arcs.*

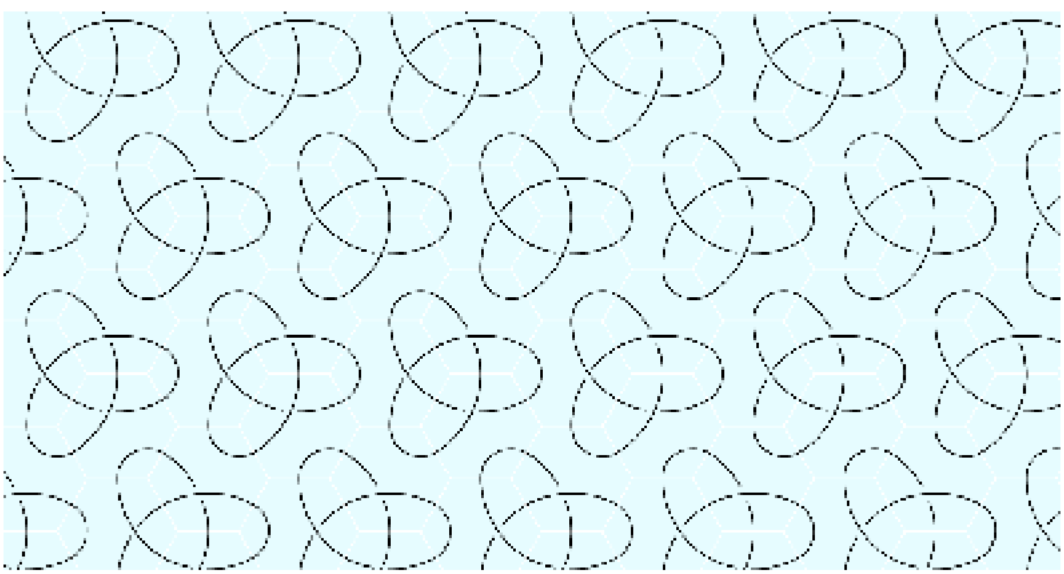

Figure 8 : *An example based on the tessellation by hexagons. This figure shows an example containing a repeating pattern of trefoil knots based on a tessellation by hexagons (6,6,6) where each hexagon is decorated with three arcs.*

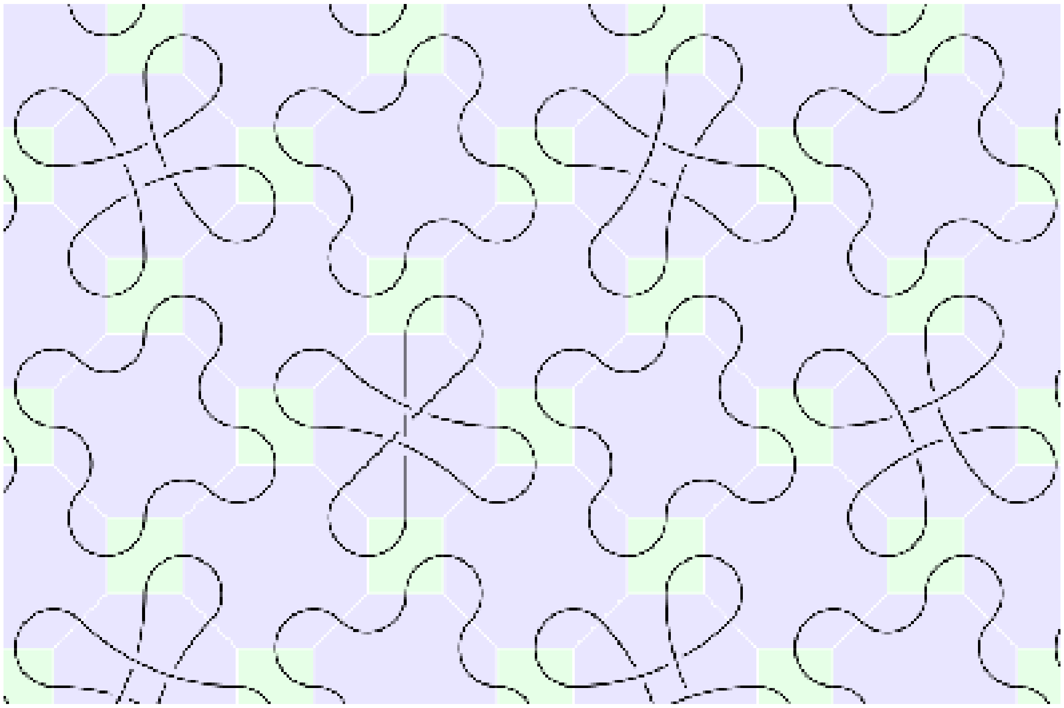

Figure 9 : *An example based on the tessellation by octagons and squares. This figure shows an example containing a several knot pattern based on a tessellation by octagons and squares (8,8,4) where each polygon is decorated with arcs connecting edge midpoints.*

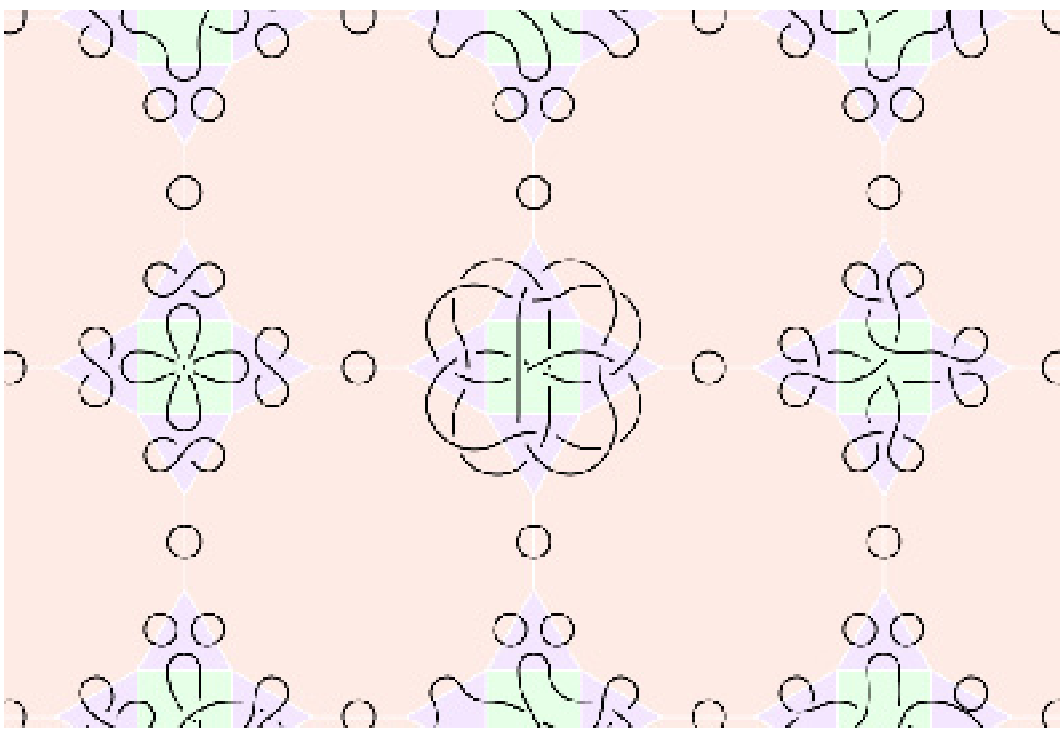

Figure 10 : *An example based on a 2-uniform tessellation. This figure shows an example containing a several knot pattern based on a 2-uniform tessellation by dodecagons, squares, and triangles, (12,12,3;12,3,4,3) where each polygon is decorated with arcs connecting edge tri-points.*

Visualizing the Roots of Complex Polynomials With Complex Exponents

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Abstract

Polynomials are often used to produce algorithmic art. In this paper I explore polynomials with complex integer exponents. Introducing complex exponents changes the number of roots, the locations of those roots and the symmetry relations (if any) among the roots. Complex polynomials with complex integer exponents [illegible] used to locate the roots of such polynomials. I obtain images by iterating the Newton Map of complex polynomials with complex exponents in order to explore the behavior of these polynomials.

Introduction

[illegible] Numerous authors have also applied the Newton iteration function, or Newton Map, to functions of a complex variable to obtain images. For example, see Crass [2] , Gilbert [3], Kalantari [5], or Sahara and Diellit [6]. This paper extends that body of work to polynomials with complex exponents. I start with polynomials of the form $z^{\lambda} - \rho = 0$, $\lambda = n + mi$ where n, m are integers and i [illegible] z a complex variable. I then consider complex polynomials with complex exponents involving several terms.

Each point in the complex plane is assigned a color based on which root that point converges to (if it converges at all). This allows one to visualize the behavior of these functions as well as the number and location of the roots of these polynomials.

[illegible] exponents. I then discuss the differences in the the number of roots, the locations of those roots and the symmetry relations (if any) among the roots that occur when one introduces the possibility of complex integer exponents. Finally I present some of the images I have obtained by applying Newton Maps to complex polynomials with complex exponents.

Newtons Method for finding Roots of NonLinear Equations

[illegible] straightforward. Start with an initial guess for a root of $f(z) = 0$ [illegible]

$z_{i+1} = z_i - f(z_i) / f'(z_i)$ where $f'(z)$ is the derivative of $f(z)$ and stop when $|z_{i+1} - z_i| \leq \varepsilon$.

It is well known that each root of $f(z) = 0$ is an attracting fixed point of the Newton Map $z_{i+1} = z_i - f(z_i) / f'(z_i)$. In the case of the simplest polynomial, $z^n - \rho$, the Newton map would be the iterated function

$$z_i = \frac{(n-1)z_{i-1}^n + \rho}{nz_{i-1}^{n-1}}$$

Solutions of Complex Polynomials
with Complex Exponents

Most, but not all, prior work obtaining images using Newton Maps has focused on complex polynomials [illegible] with complex exponents.

The value of a complex variable raised to a complex power can be computed by using the relationship

$$z^i = e^{i\ln z} = e^{-\theta} e^{i\ln r} \qquad \text{where} \quad z = re^{i\theta}$$

See Churchill and Brown [1].

In order to examine the changes that occur when one introduces integer complex exponents, consider the simple case of

$$z^\lambda - \rho e^{i\phi} = 0 \quad \text{where} \quad \lambda = n + mi,\ z = re^{i\theta}.$$

The solutions to this equation are:

$$\ln r = \frac{n \ln \rho + m(\phi + 2\pi j)}{n^2 + m^2} \qquad \text{for integer } j \text{ satisfying}$$

$$\theta = \frac{n(\phi + 2\pi j) - m \ln \rho}{n^2 + m^2} \qquad -\pi < \theta < \pi$$

The properties of the solutions to $z^{\lambda} - \rho e^{i\phi} = 0$ include:

• If $n, m > 0$ then the number of solutions is either $2s$, $2s+1$, or $2s+2$ where s is the greatest integer in $(n^2 + m^2)/2n$.

• If m=0 (integer exponents), it is well known that all of the solutions lie on a circle of radius $\rho^{1/n}$. If $n, m > 0$ all of the solutions lie on a logarithmic spiral given by

$$\ln r = \frac{m}{n}\theta + \frac{(n+m)\ln \rho}{n^2 + m^2}$$

• If $n = 0$ there are a countably infinite number of solutions if

$$-\pi < {}^{\ln\rho}\!/_{m} < \pi$$

otherwise there are no solutions in the complex plane.

Symmetry Relationships

The Newton Map for $z^n - \rho e^{i\phi} = 0$ is invariant with respect to rotation by an angle of $2\pi / n$. The resulting image will have rotational symmetry. If ϕ is zero or minus π, the Newton Map will also exhibit a mirror symmetry.

The invariance or symmetry operation for the Newton Map of $z^{n+mi} - pe^{i\phi}$ is more complicated. The Newton map will be invariant with respect to a rotation by an angle equal to $2n\pi / (n^2 + m^2)$ followed by multiplication by $\exp(2\pi m / (n^2 + m^2))$.

Some implications for images obtained from Newton Maps of complex polynomials with complex exponents include:

- Images will exhibit spiral-like characteristics
- Images may exhibit a discontinuity along negative real axis

- Convergence may be problematic

- If the real part of the exponent is zero, there are infinitely many attracting fixed points in any neighborhood of infinity. This contrasts with the well known fact that if the imaginary part of the exponent is zero, infinity is a repelling fixed point.

Coloring Algorithm for Images

I have divided the images in this paper into two groups. The first set of images are obtained from simple equations of the form $z^{n+mi} - \rho e^{i\phi} = 0$ and are referred to as pinwheel images. The second set of images use polynomials with more terms and have a circular structure. They are referred to as circular-type images.

The coloring algorithm I use is as follows. For each point z in a pre-determined grid, apply the Newton map. If the equation has j roots, choose $k \le j$ roots and $k+3$ colors. Color each point based on the root to which root it converges. The additional three colors are for points that converge to a root not in the k roots, do not converge, or become unbounded.

In all cases I have shown the polynomial used to produce the image beneath the image. In almost all cases I have applied more than one color scheme to the same image.

Pinwheel Images

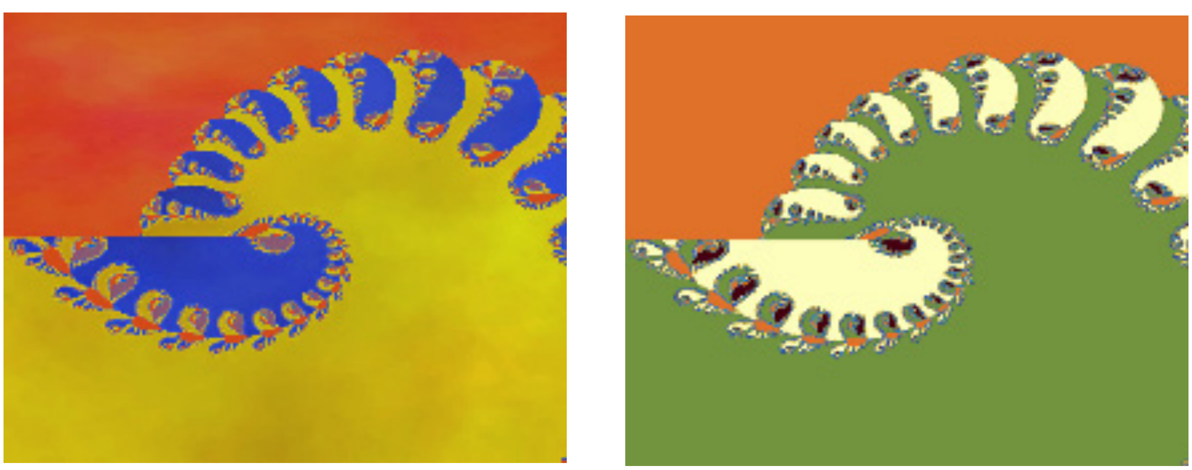


Figure 1 $z^{2+3i} = 1$ using two color schemes

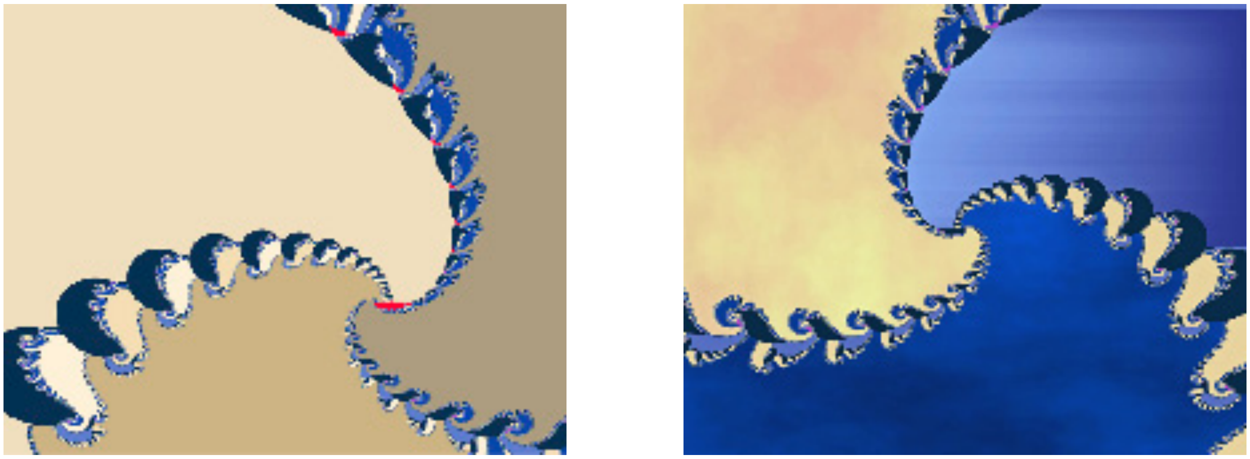


Figure 2 $z^{3+2i} = i = 0$ using two color schemes and orientations

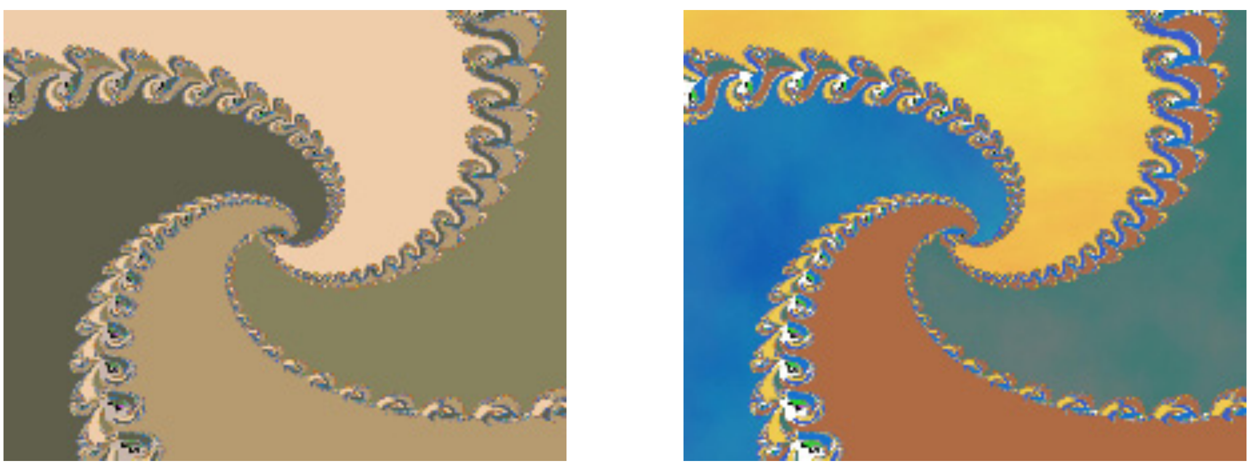


Figure 3 $z^{4+4i} - i = 0$ using two color schemes

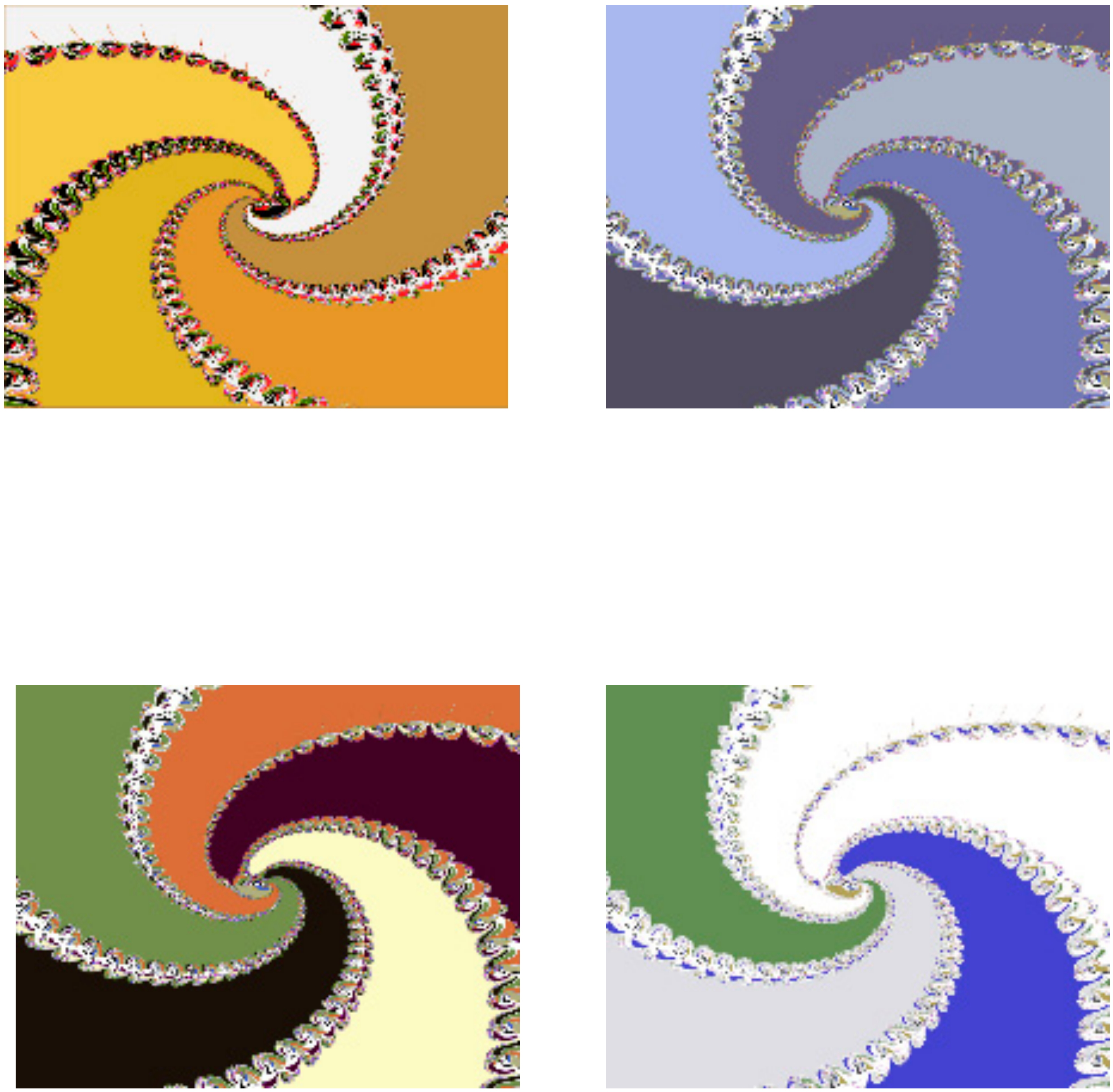
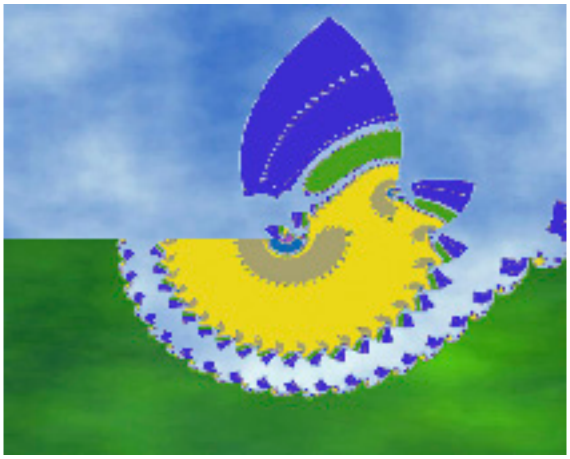


Figure 4 $z^{5+6i} - i = 0$ using four color schemes and two orientations

Circular Type Images



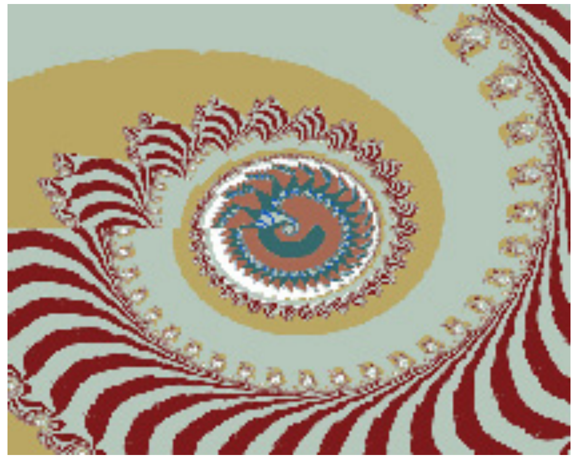


Figure 5 $z^{6i} - 2z^{2i} - 2 = 0$

Figure 6 $z^{6i} - 1.045z^{4i} + .045z^{2i} - 8.23x10^{-5} = 0$

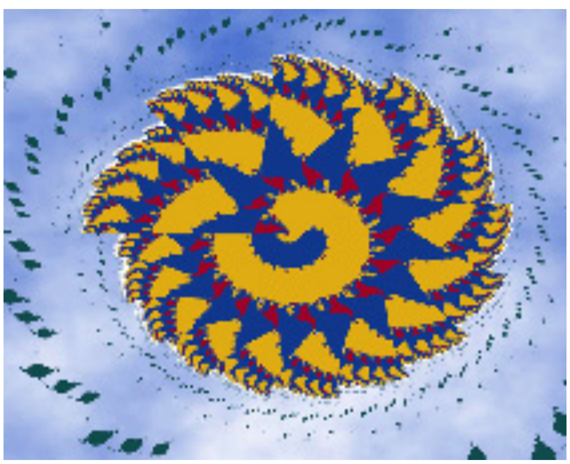


Figure 7 $z^{3i} - 1.251z^{2i} + .261z^{i} - .009 = 0$ using two color schemes, orientations and cropping.

Figure 8 $z^{6+8i} - z^{3+4i} - e^{-8\pi/3} z^{3+4i} - e^{-8\pi/3} = 0$ using two color schemes

Conclusion

[illegible]

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[illegible]

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Curved Plane Sculpture: Squares

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Abstract

Basic geometric shapes offer interesting possibilities for artistic expression. This paper considers two-dimensional planar surfaces--squares and their subdivision as triangles--that are developed as three dimensional forms. The fabrication process for the sculptures is similar to non-computational origami: a pattern on a flat surface is manipulated manually to become a three-dimensional object. In these cases, fiberglas grid surfaces are curved to become three-dimensional sculptures. They are presented as sculptural reliefs (also known as sculptural projections).

Introduction

To a geometer, a square is a quadrilateral figure which is both equilateral and right angled [1]. To the fine artist, squares and their subdivisions as triangles are common shapes to be explored and transformed into an artwork that represents personal expression and has a unique presence.

In this paper I discuss three sculptures I fabricated from a non-traditional sculptural material, fiberglas screening. All of the steps to transform the screening from a flat (two-dimensional) plane to a curved (three-dimensional) plane were done by hand.

Fiberglas screening, a flexible planar surface, responds to warping, curving, and torquing. The sculptural form is moveable and changeable. Gravity and tension affect it. Volumes alternate with transparency. [illegible] an appreciation of curved space.

Light and Shadows

Light allows the viewer to see the form. Light is a major contributor to the sculptural visualization of a mathematical idea. It is not necessary to have a light source of specific direction or intensity to understand the nuances of a geometric figure. However, the visual experience of three dimensional sculptural forms depends upon it. A shift in the location or intensity of the light source, or multiple light sources, dramatically changes the perception of the sculpture.

Light also presents shadows of varying intensity so that the viewer knows two things: the artwork is not flat and the artwork has volume. For comparison, two of the sculptures described in this paper are photographed both with a non-directional and a directional light source. The latter also enhances the perception of the moire patterns.

Sculptures Based on a Square

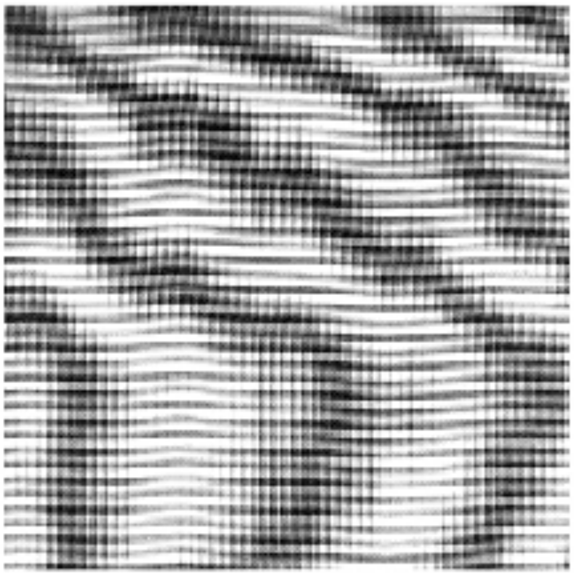

Figure 1: A quare s

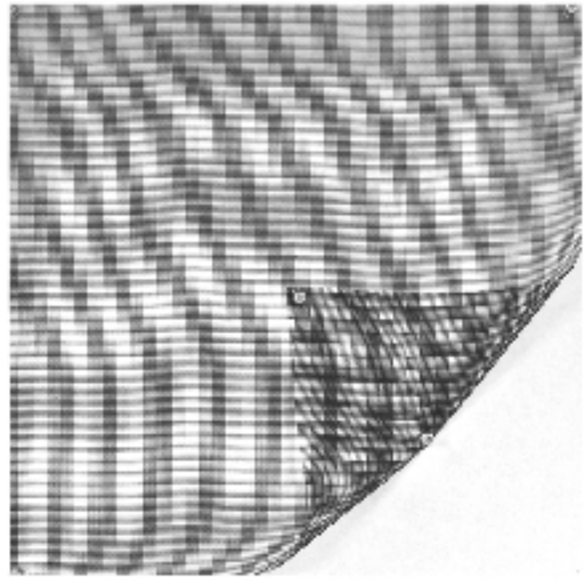

Figure 2: Step 1

Square Pattern 1. Figure 1 shows the material for a sculpture. It is a fiberglas screen cut in the shape of a square. The square screen is a grid of many small squares. Figure 2 shows the first step in constructing a form. The lower right vertex is folded toward the center of the square. In origami, this would be known as a valley fold. However, unlike origami, the fold is kept soft--rather than creased or flattened--to emphasize the curved squares. The first of many moire patterns is now evident.

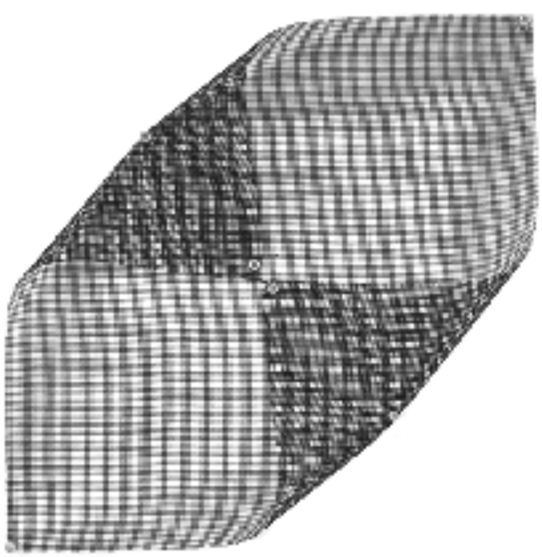

Figure 3: Step 2

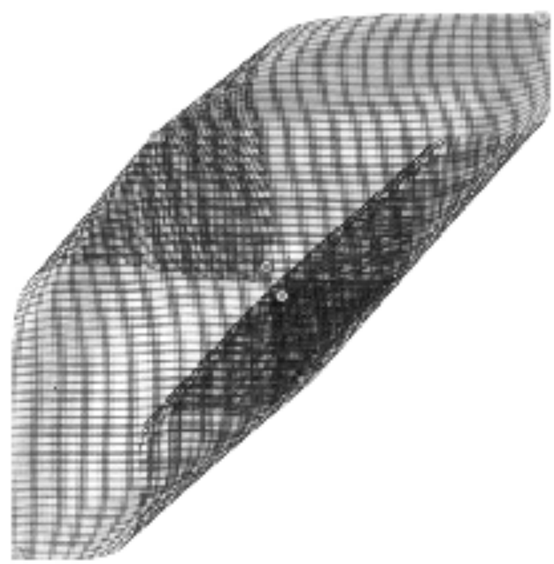

Figure 4: Step 3

Figure 3 shows Step 2, where the opposite vertex is folded toward the middle. Figure 4, Step 3, shows the edge of the fold from Step 1 folded toward the middle. The multiple layers are darker and dense moire patterns appear.

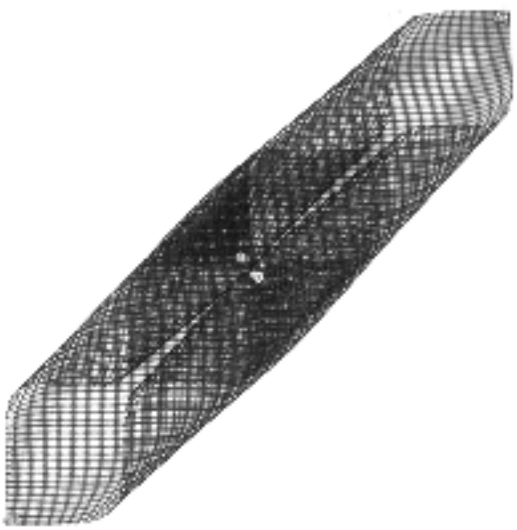

Figure 5: tep 4 S

Figure 6: Step 5

Figure 5 shows Step 4, where the other edge formed in Step 2 is folded toward the middle. An imaginary axis connecting the two unfolded remaining vertices reveals the bilateral symmetry of the pattern at this step. In Figure 6 the form bends back from the middle upon itself. The edges from Steps 3 and 4 are folded together. In origami, this would be known as a mountain fold.

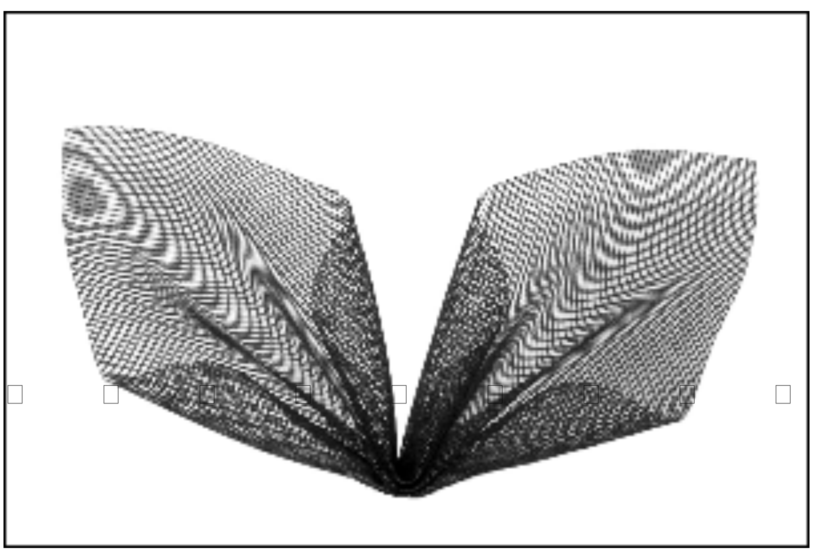

Figure 7: *Curved Squares 3*,

Figure 7 illustrates Step 5; the shape is folded in half, allowed to expand from that center fold, and viewed -964 ;, :0 #, 9, :3 ;05. -694 0 (;;(, ;6 (5 (9;0 ; : (5(: 7(05;, 0;, (5 79, :, 5;, (: (

sculptural relief, projecting outward with more depth and more volume than a bas-relief. The lighting is non-directional. I drew a square with a pencil line on the canvas to refer viewers back to the original geometric shape of the form. Titled *Curved Squares 3*, this form is also the unit form for *Curved Squares 1*, illustrated below.

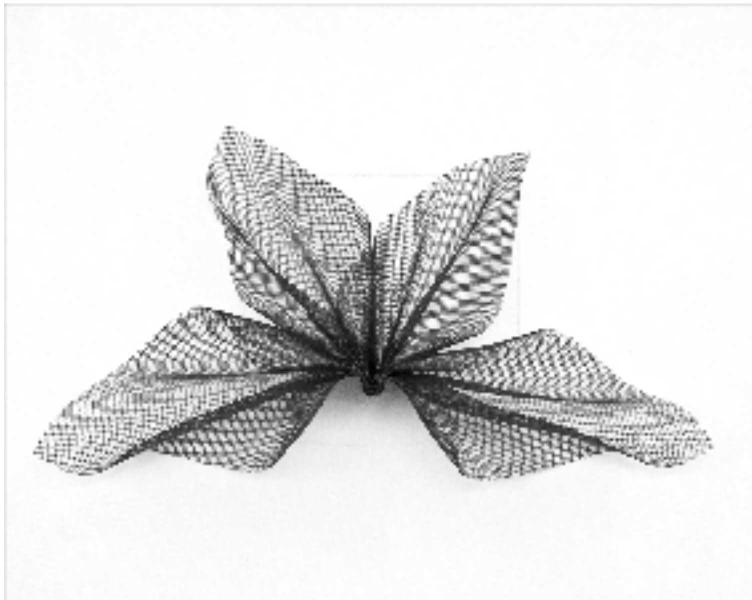

Figure 8: *Curved Squares 1,* [illegible]

Figure 8 is constructed by joining two of the unit forms shown in Step 5. The resulting form is attached [illegible] and more volume than a bas-relief. The lighting is non-directional. I drew a square with a pencil line on the canvas to refer viewers back to the original geometric shape of the form.

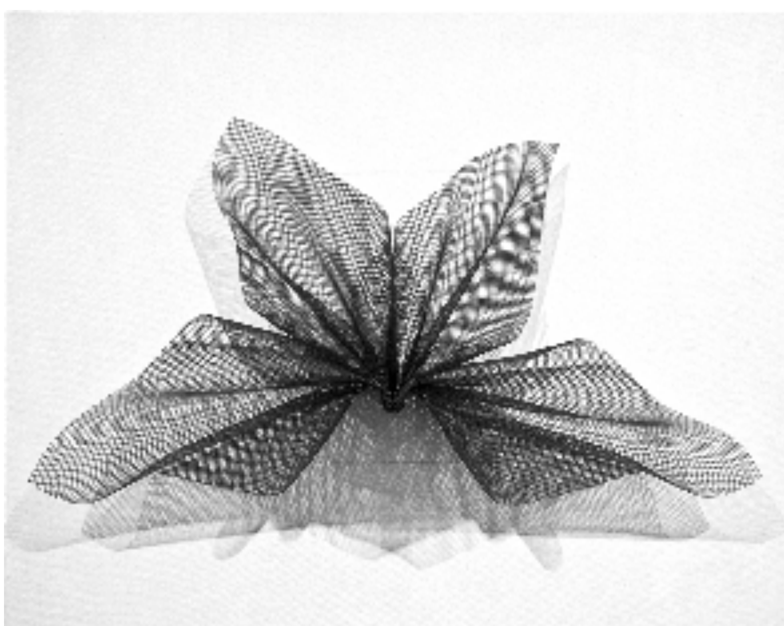

Figure 9: The sculpture in Figure 8, *Curved Squares 1*, with a directional lighting source.

In Figure 9 the cast shadows give more information about the form and increase the complexity of the visual experience. They preserve the moire patterns.

Sculptures Based on a Triangular Subdivision of a Square

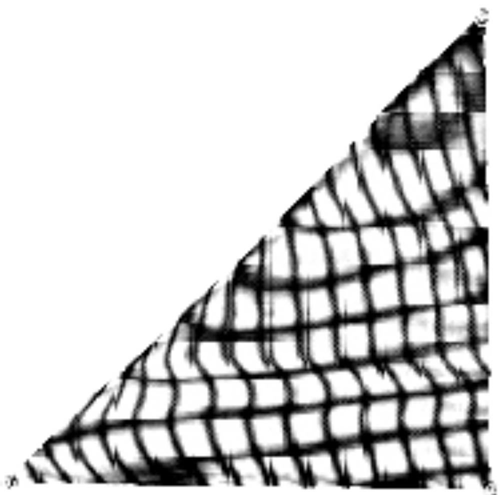

Figure 10: Step 1

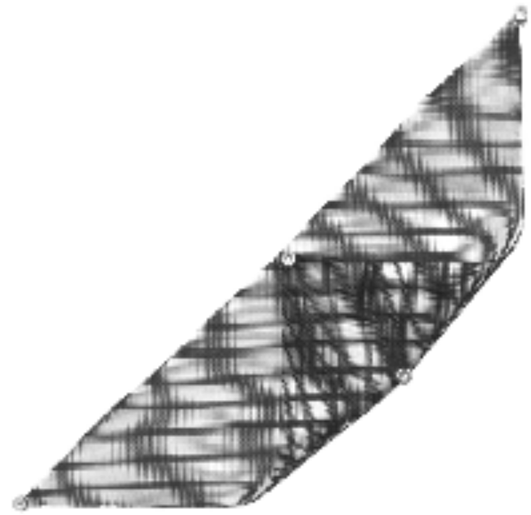

Figure 11: Step 2

Figure 10 shows the first step in constructing a form from a triangular subdivision of a square. The manual process is similar to Sculptures Based on a Square discussed above. In Figure 11, Step 2 shows how the lower right vertex is folded toward the center of the triangle.

Figure 12: Step 3

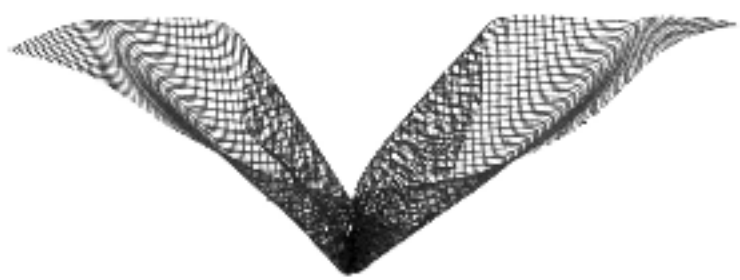

Figure 13: Step 4

Figure 12, above, shows Step 3, where the edge formed in Step 2 is turned toward the middle. In Figure 13, the shape is folded in half, allowed to expand from the center fold, and viewed from the side. This form is the unit form for *Curved Squares 2*, illustrated below. It can also be a standalone sculptural form.

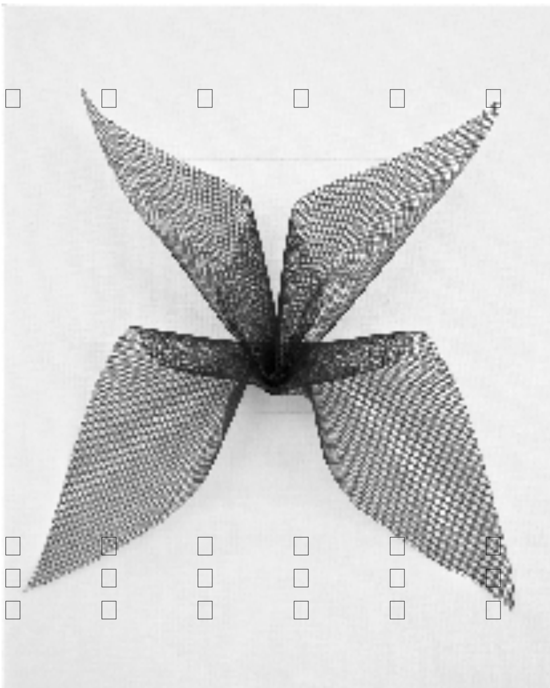

Figure 14: *Curved Squares 2*,

more depth and more volume than a bas-relief. The lighting is directional.

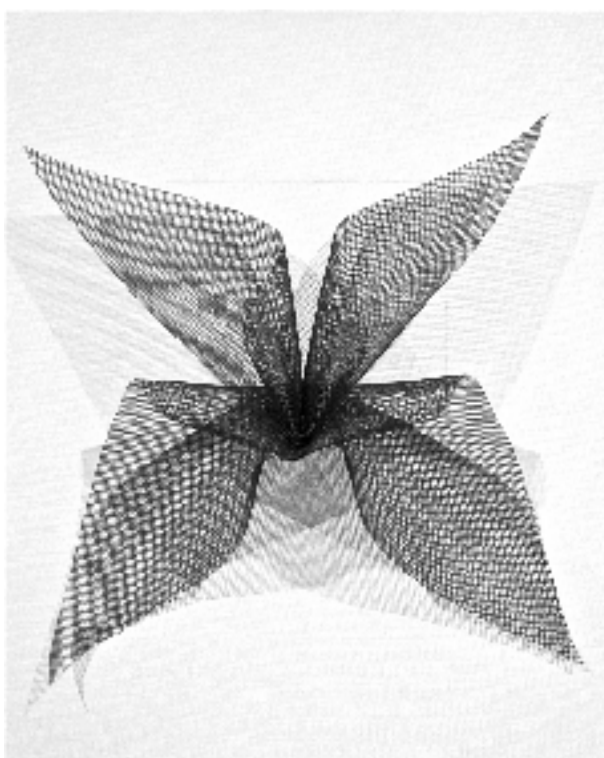

Figure 15:

Aesthetic Comments

These sculptural forms provide a path for the eye where there was none before. They are guides for a perceptual voyage of the eye and of the mind. When we look, we visually enter the spaces of the sculptures and respond to the geometric energy of the squares curved in space. Each square is an integral part of the piece. If we stand close, we can see each one in detail; if we stand back, we see how, massed as a grid, their flexibility has formed a stable structure. As with all artworks, we can reflect on their power in our own lives.

Results

Artistically, squares and their subdivision as triangles are interesting subject matter for curved sculptures. By using simple folding patterns and choosing an interesting material, a variety of sculptures can be created to change our perceptions of spatial dimensions.

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Fractal Decomposition as Building Type: The New Buildings of New Levittown

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Abstract

A technique derived partly out of recent work with V-variable fractals is used to generate a building type for an urban scaled redevelopment project for a suburban downtown. The evolution of concept and technique are described along with the overall design. A brief discussion of the application of mathematics to architecture and how it informs my work in general is touched upon.

Background

New Levittown is a speculative downtown redevelopment project originally submitted for the 'Build a Better Burb' competition held in 2010 [1]. The group that organized the competition, the scope of which included all of Long Island, New York, cited the dire need for housing along with the more intangible need of creating reasons for staying in Long Island. Contestants were invited to select a site and develop a program that addressed the goals of the competition. I chose the city of Levittown primarily because of its remarkable history as America's paradigmatic suburb. The introduction of assembly line methods to the housing industry resulted in over 17,000 homes built in a three year period starting in 1948 (figure 1, left). The style chosen for the homes, Cape Cod, came in five variations, and owners were encouraged to make their own improvements. The homes were sited with different orientations on their lots, avoiding the potential monotony of seeing nothing but the same house and the same view of it. The downtown displays the flipside of the regularity of these homes, and is characterized by a spontaneous (that is unplanned from a long term point of view) development pattern dominated by single story buildings and asphalt (figure 1, right). The sprawl of Levittown's downtown is as much a part of the landscape of suburbia as are the repetitive homes.

Figure 1: *Left, view of Levittown circa 1949; right, view of downtown present day.*

At first glance the disorganized sprawl of the strip is utterly spontaneous, and without order. One way to address this would be to start over and demolish everything that gets in the way of a new design. Another way would be to dress it up and build additional elements that respect each building's specific disposition (its geometry/footprint, scale, idiom, etc.). A third way would be to overlay onto the spontaneous irregularity of the strip an utterly foreign design, and then relying on the accidents created by their mash-up to engender specific conditions not reducible to idiom or design. I chose this way because I felt that it would be a good way to introduce a new program to the site while also embracing the existing conditions, both the sprawl and the Cape Cod landscape that engulfs the downtown, in effect letting the site 'read through' the new intervention.

I based the imported system off of a concept featured in the urban plans for New Frankfurt by Ernst May. Zeilenbau refers to a planning principle using a serial pattern of bar-like buildings. Numerous settlements were built in Germany in the 1920's and 30's under this principle [2]. That this would set up a regularity there was no doubt, but I also felt that it resonated, in an abstract sense, with some of the minimalist art that came later in the century. What I liked about this possible resonance was the aspect of the user's position with regard to the object that completed it, activating it and the space around it. I was also quite intrigued by some of the tricks used in New Frankfurt to introduce variation into the serial system. In one project the ends of the bars were turned at right angles, creating a kind of boundary around the development. In another case, the bars were offset at various points to create irregularities in otherwise repetitive sightlines. With this stew of thoughts I imagined a kind of clinamen whereby these various elements would collide with each other as they fell through space on their way to the ground.

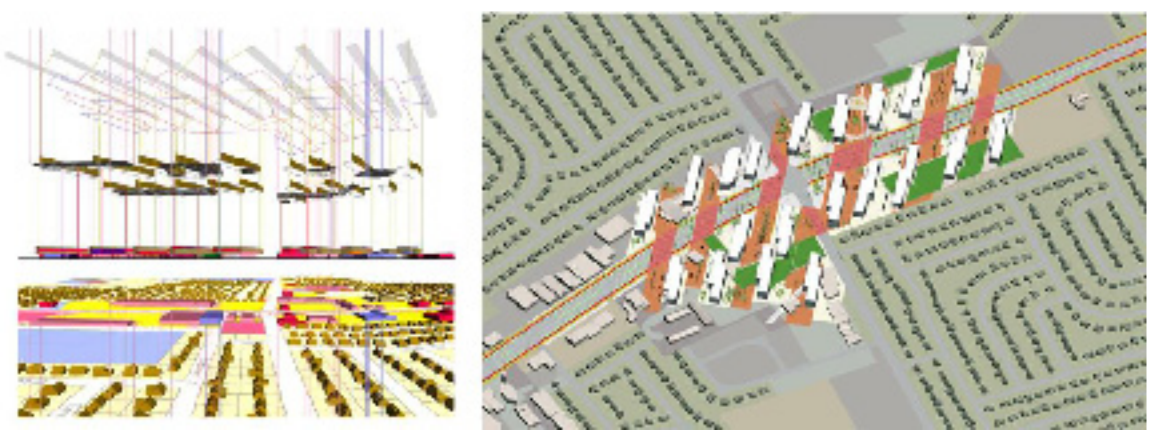

Figure 2: *Left, project clinamen; right, arial view of New Levittown*

The design for the urban plan of New Levittown resulted from this thought experiment (Figure 2). The bars become five story mixed use structures housing a community college, dormitories, office space, apartments, and commercial spaces. Existing malls and other buildings are renovated and rebuilt and allowed to intersect the bars. This created an irregular roof datum that meanders through and between the bars. The ground level became an amorphous public space doubling as a parking lot. Spontaneity permeated regularity.

The competition entry focused on urban issues. After it was over I went back to work out the design of the buildings which I always knew were crucial for the work as a whole. A while back when I was doing projects with megastructures I realized that what I liked about them was that you can add or subtract to them without changing them. Therefore the architecture lies not in its outline or form or shape, it's more about connection, structure and the interplay between sameness and variation. I realized that here with New Levittown I was working with a megastructure, or better under the influence of this insight about megastructures [3, 4]. I was intrigued by what you could get away with doing to the bars that also preserved (or created?) a deeper unity, one that couldn't be touched by manipulating the parameters used to generate changes. The work of one of Le Corbusier's assistants came to mind. Better known as a modern composer, Xenakis' music was very familiar to me and I knew he used Markov Chains as a way of exploiting stochastic processes while being guaranteed a kind of overall unity in the work as it evolved [5]. Markov chains are iterative systems that converge to a specific behavior and are like fractals in that regard. I quickly was reminded that fractals are constructed by affine transformations, a highly suggestive "mathematical material" for architectural application. I wondered if the Zeilenbau bar logic of the new buildings of New Levittown could be enriched by a mathematical exploration along these lines.

Why Fractals

First of all, what's amazing about a fractal is its simplicity- they are maps that take on their solutions as inputs whose functions are typically standard affine transformations. They have an organic quality that is not based on analogy or aesthetic imitation, consequently they are used to model extremely complex and detailed aspects of nature. I was familiar with two kinds of maps that generate fractals, L-Systems and Iterated Function Systems (IFS). The former had an interesting branching aesthetic, but was probably not going to be as fruitful for me as the latter. An IFS typically operates on $\mathbb{R}^d$ or more generally on a compact metric space $(\mathbb{X}, d)$, to produce a fractal set attractor. The set S is composed of a number of components or elements, where each is a scaled image of itself, and each of these elements is composed of the same number of images of itself similarly scaled, and so on. The functions f_n^m that scale and place the images must be contractive maps. Generally speaking then, the functions of an IFS are a kind of affine transformation called similitudes, i.e. composed of the congruency preserving transformation (rotations, translations, reflections) and a fixed ratio scaling transformation. Generally there are a number of fixed points and a function associated with each one, so if there are three vertices A_1, A_2, A_3 as in a Sierpinski Triangle (figure 3), then f_1, f_2, f_3 are corresponding contraction maps, and the set S is made up of the subcomponents $f_1(S)$, $f_2(S)$ and $f_3(S)$: $S = f_1(S) \bigcup f_2(S) \bigcup f_3(S)$. The set of the three maps $F = (f_1, f_2, f_3)$ is called an Iterative Function System (IFS). It doesn't matter what you begin with, that is starting out with any compact set T one defines $F(T) = f_1(T) \bigcup f_2(T) \bigcup f_3(T)$ and if continued, i.e. for k>=1 $T_k = F(T_{k-1})$, T_k converges to S as k>infinity. Thus, S is called the fractal set attractor of the IFS F. This method of getting to S is called the convergent or backward process [6].

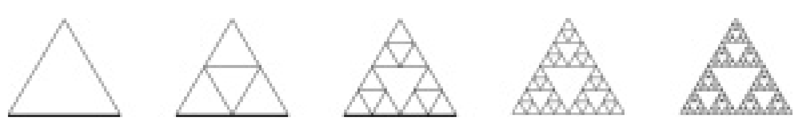

Figure 3: *Sierpinski Triangle*

There's another way to converge to S, the forward or chaotic process, sometimes referred to as the Chaos game. Here one begins from any point and recursively defines $xk = f_{,}(x_{k-1})$ where f is chosen independently and with a prescribed probability.

Figure 4: *V-var fractal*

The process is ergodic and converges to the attractor *S*. The chaos game intrigued me in the sense that it opened the possibility of manipulating the input set. From the idea of V-variable fractals of Michael Barnsley [7], there could be several inputs, and there could be multiple functions that could operate probabilistically on an input. Figure 4 shows one such fractal. The first level at n=1 shows the input buffer set V as three equilateral triangles. To generate the next level, one out of two functions is chosen with equal probability (this could be weighted) to operate on an input buffer, the difference being whether to scale by 1/2 or 1/3.

Application

I proceeded with the idea of the Chaos Game but quickly realized two important limitations, 1) a rectangle and four points will not produce a fractal set because it fills in the unit square, and 2) a fractal is dependent on scaling, which is of questionable utility for architecture- is it important for elements of architecture to have the property of being self-similar at different scales? So, aware that I may not be working strictly with fractals I began by writing LISP routines that played around with the formulas that produced them.

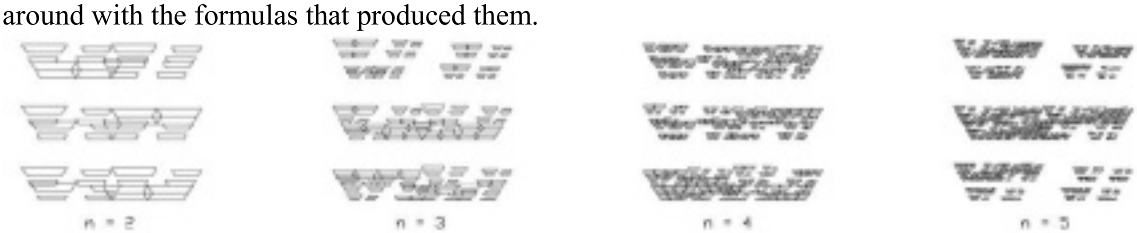

Figure 5: *IFS with trapezoidal input buffer*

First, I tried manipulating a V-var IFS by using four fixed points instead of three and using a trapezoid as input buffer to avoid the first limitation mentioned above (figure 5). Next, I turned the input set into a space curve (figure 6).

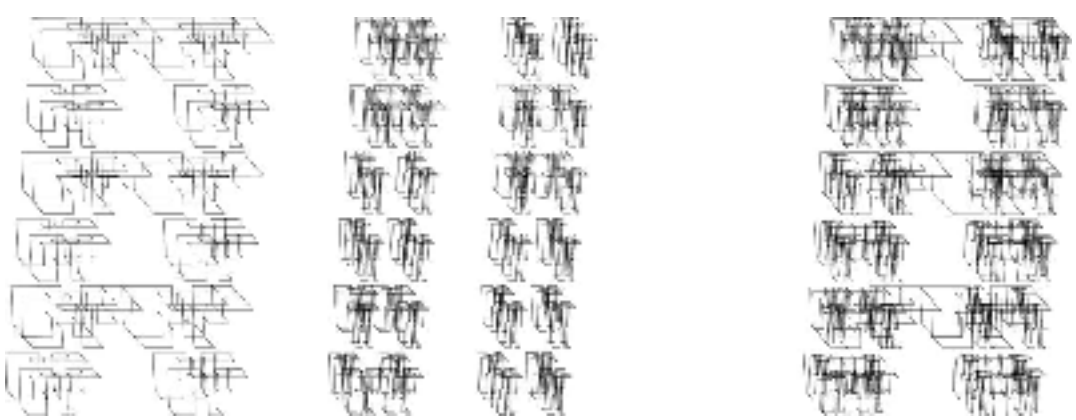

Figure 6: *Left, two levels of an IFS with space curve as input buffer; right, two levels superimposed.*

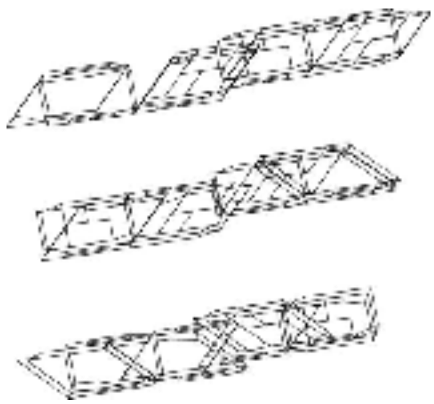

Figure 7: *IFS with room-like buffer*

Next, I turned the input set into a room (Figure 7). Taking a cue from V-var IFS use of multiple inputs and the random aspect of the Chaos game, I began to think of a polygon or three dimensional simplex that transformed or even permuted in some probabilistic way.

But I became displeased with the direction things were going at this point. It became clear to me that I could continue to generate objects of great intricacy that may even become interesting, in and of themselves. The problem was that I felt I was losing touch with the site. So I backed up a bit and re-examined premises. What about the fact that the algorithm only breaks things down into smaller bits? What if we say that we want a part to map to another part of itself- in that way the region outside the part enlarges instead of breaks down between levels. The way to do this is to have an input figure that would map onto itself after the transformations are applied. I realized that for a part to map onto a part either the contractive map would produce overlaps or I could superimpose levels and let them accumulate at each level. I went with the latter idea. I also felt that it mustn't be every case of the mapping that a colinearity results, so employing the random aspect, I restricted the possible cases of overlap by having there be permutations where this did not occur. I then observed that I only needed two levels to enact this. I began to formalize this idea. I decided to first deal with an overlap of vertical elements. I borrowed Michael Barnsley's scalars for his V-variable fractals- 1/2 and 1/3. The problem is defined explicitly as follows and is diagrammed in figure 8.

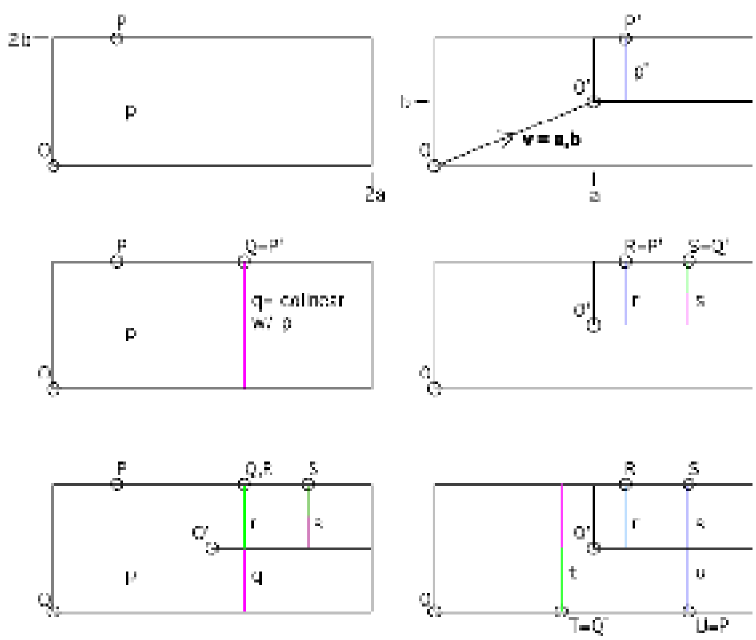

Figure 8: *Visual diagram of colinearity objectives between input and outputs of an IFS system*

Problem: Find a point P such that a line passing through it has an image under a given transformation matrix **M1** that is colinear to another line through Q and has an image under a different transformation matrix **M2** that is colinear to the image of Q under **M1**.

First we derive **M1.** Breaking this up, first we find the transformation matrix sending P to Q:

$$1)\quad \begin{vmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1/2 & 0 & a \\ 0 & 1/2 & b \\ 0 & 0 & 1 \end{vmatrix}$$

Then the transformation matrix sending Q to S (note that it is the same thing as applying the above transformation twice to the same point):

$$\mathbf{M1} = \begin{vmatrix} 1/2 & 0 & a \\ 0 & 1/2 & b \\ 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1/2 & 0 & a \\ 0 & 1/2 & b \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1/4 & 0 & 3/2a \\ 0 & 1/4 & 3/2b \\ 0 & 0 & 1 \end{vmatrix}$$

Next we derive **M2**, the transformation matrix sending P to U:

$$\mathbf{M2} = \begin{vmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 2a \\ 0 & -1 & 2b \\ 0 & 0 & 1 \end{vmatrix}$$

Solution: send a point P at x, $y = 4$, $z = 1$ through **M1** and **M2** and set them equal to each other (note that I'm just picking 4 at random, doesn't have to be equal to a point on the box and that's OK since we are interested in the vertical lines that pass through point P.)

$$\mathbf{M1}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = \begin{vmatrix} 1/4 & 0 & 3/2a \\ 0 & 1/4 & 3/2b \\ 0 & 0 & 1 \end{vmatrix} \times \begin{matrix} x \\ 4 \\ 1 \end{matrix} = \begin{matrix} x' = 1/4x + 3/2a \\ y' = 1 + 3/2b \\ z' = 1 \end{matrix}$$

$$\mathbf{M2}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = \begin{vmatrix} -1 & 0 & 2a \\ 0 & -1 & 2b \\ 0 & 0 & 1 \end{vmatrix} \times \begin{matrix} x \\ 4 \\ 1 \end{matrix} = \begin{matrix} x' = -1x + 2a \\ y' = -4 + 2b \\ z' = 1 \end{matrix}$$

Looking only at the equations for *x'* we have:

2) $\frac{1}{4}x + \frac{3}{2}a = -1x + 2a$, which reduces down to

3) $x = \frac{2}{5}a$, or $\frac{1}{5}$ of 2a which is the overall length of the rectangle.

So we've found P, to find Q send the point $\frac{2}{5}a$, 4, 1 through 1) and the x coordinate is 6/5a or 3/5 of 2a. So, thinking only in two dimensions at this point, if the elevation was divided by 5ths, then when scaled by one half an overlap can occur between iterations (levels).

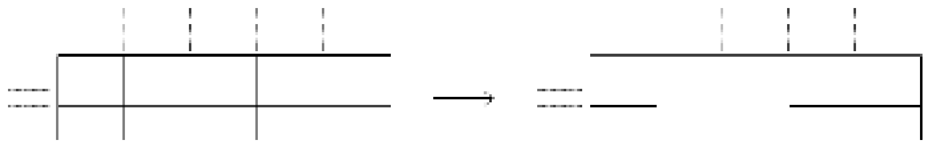

Figure 9: *input buffer formed around fifths*

In addition, if scaled by 1/3 without rotation, the line through point Q could be colinear with the line through P. So there are colinearities with both the 1/2 and 1/3 transforms. What was even more interesting was that the result came out to be not only a nice integer, but also match the number of floors I had intended on using, five. Thus both vertical and horizontal elements could easily be set up according to a module that would produce an overlap between levels according to certain transformations. It was because of this coincidence that I stayed with these transforms even though they were chosen more or less out of convenience. Surprisingly, the 1/2 scale and the 1/3 scale choices turned out to be natural fits for an elevation divided by 5ths. The input set was drawn as lines on these intervals (figure 9).

Interior Relations

The overall geometry of a typical bar was developed in the following way. I began in two dimensions using a geometric tool I stumbled upon that converts 5ths into 4ths. Starting with the bar in plan, a rectangle measuring 16 by 68 meters, two lines connect points at 2/5 and 4/5 on one side to points at 1/5 and 3/5 on the other side. Where these lines intersect with the body diagonal determines the column locations and the number of bays, 12. The alternating columns are spaced out at about 5.7 meters apart in the long direction but have a periodic spacing in the short direction (figure 10). Now I had geometric raw material to make more complex input buffers. I could use the 5ths to 4ths diagram to create a three dimensional simplex that relates insides to outsides. The simplex, as shown in figure 11, permutes in four ways. There's one iteration overall, and four output buffers are nested within the input buffer using the corners of the rectangular boundary of the input buffer as the fixed points for the transformations. The functions for each fixed point come in pairs, distinguished by 1/2 or 1/3 scaling, and are applied with equal probability.

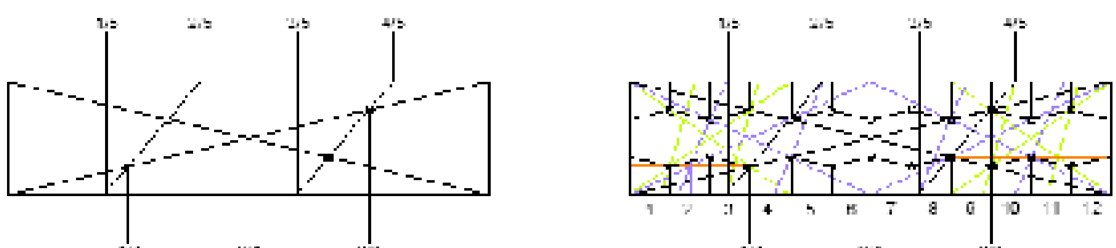

Figure 10. *left, geometry that converts 5ths to 4ths, right, column bays*

The reason for doing this is to connect interior to exterior. The geometries and rhythms of the two work together to create a spatial tension. This falls under a technique that I have been calling Decomposition. This is a technical term that comes from the act of taking a known entity and seeing what happens to it when you change it or look at it differently. In that sense it is work that is done on both typological and topological levels and has the effect of distributing unfamiliarity through familiar environments. This to me is what creates depth and richness in architecture, and should be seen as opposed to confining the unfamiliar, or new, under one boundary or aspect that clearly distinguishes it from the familiar space that the observer inhabits. Much of the work in contemporary architecture done under the influence of parametricism is regressive in this way. Decomposition, as I see it, is a way of operating deeply on types and unlocking the richness of sites and their architecture.

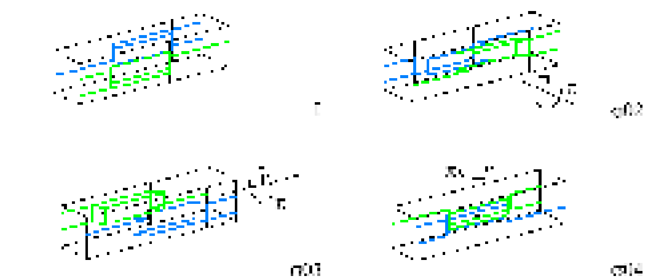

Figure 11: *permutation group of the input buffer*

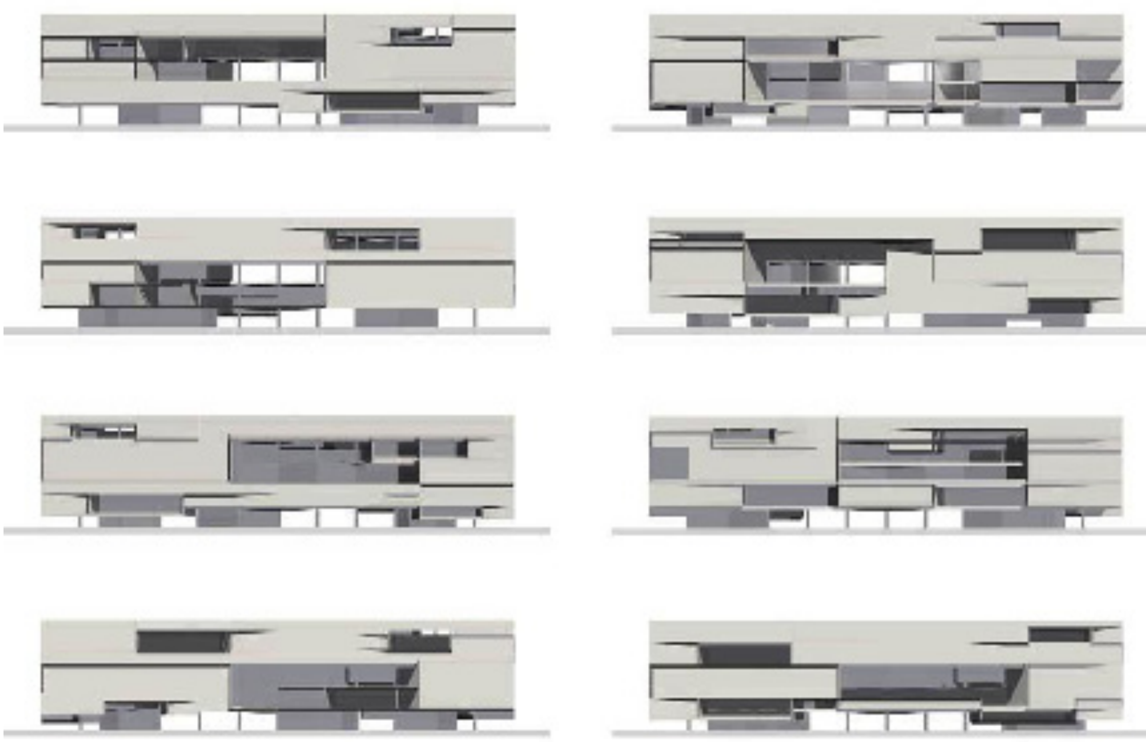

Figure 12: *Sample instances of the building type.*

The Project

The site is hatched by bands of streets, which are actually hybrids of streets and landscaped surfaces featuring semi-porous paving and xeriscapes. The former big box retail buildings are retained and outfitted with a green roof and converted into open, mixed use facilities that can handle flexible programs. These elements merge and penetrate the new buildings. The former unused plots of land and parking lots are converted into a network of landscaped spaces, rainwater gardens and groundwater retention facilities.

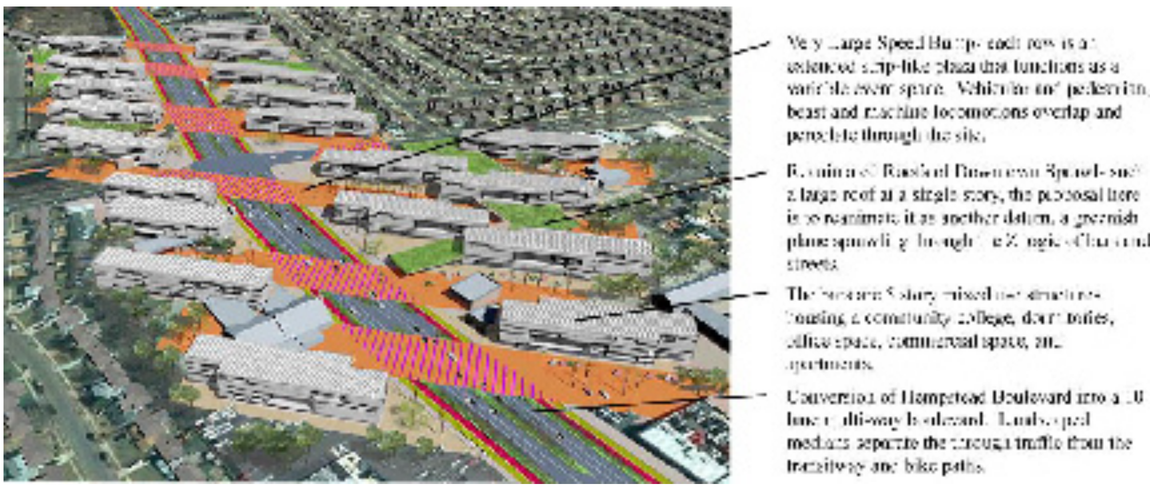

Figure 13: *Arial view of New Levittown.*

New Levittown responds to the program brief by creating a quasi-urban densification that would serve as an attractor for the area. The treatment of various existing site elements and infrastructures, namely the highways which divide the site into quarters and the plots of unused land, are seen as part of a phasing plan whereby sustainability features such as mass transit boulevard and conversion of unused plots of land into rainwater gardens and bio retention facilities grow into the design. The project is anchored by a 'modernist imaginary' projected onto a site whose pre-existing site elements are reanimated.

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Figure 14: *Rendering of one of the new buildings of New Levittown.*

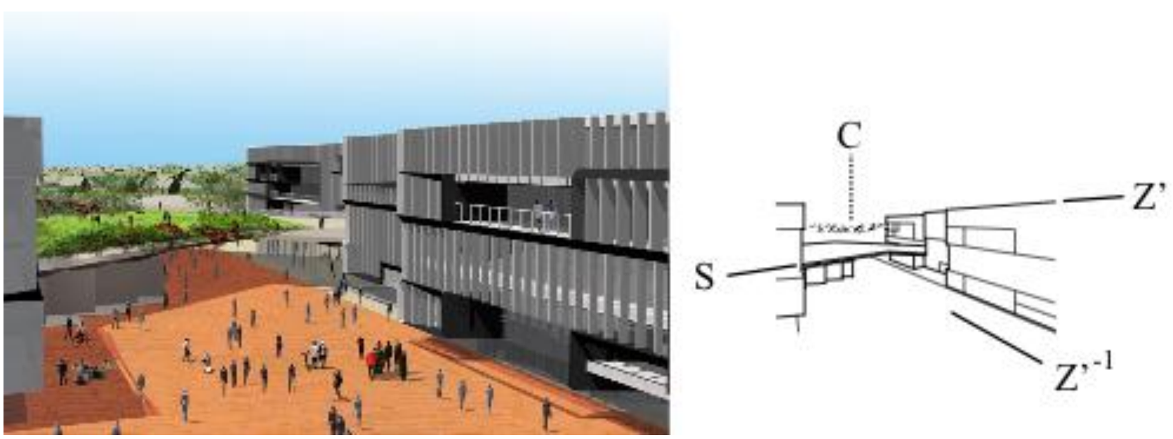

Figure 15: *Project gestalt, Z is the bar logic of the new buildings, inverse Z the "street" logic of the space between the new buildings, C are 17,000 Cape Cods that surround the downtown, S is the reanimated sprawl that meanders through the bars.*

Figure 16: *Elevational view.*

Figure 17: *Oblique view from a street.*

Figure 18: *Axial view along a street.*

CONTENTS